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BIHARMONIC SUBMANIFOLDS IN SPACE FORMS

Habilitation Thesis

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To my daughters

Abstract

The present thesis is devoted to the study of biharmonic submanifolds in real, complex and Sasakian space forms. First, we shall present some ideas that have encouraged the study of the biharmonic submanifolds and of the geometry of biharmonic maps, and then we shall describe the results gathered in the thesis.

Denote by $C^\infty(M,N)$ the space of smooth maps $\varphi:(M,g)\to(N,h)$ between two Riemannian manifolds. A map $\varphi\in C^\infty(M,N)$ is called *harmonic* if it is a critical point of the *energy functional*

$$E : C^\infty(M,N)\to\mathbb{R},\quad E(\varphi)=\frac{1}{2}\int_M |d\varphi|^2\,v_g,$$

and it is characterized by the vanishing of the tension field

$$\tau(\varphi)=\operatorname{trace}\nabla d\varphi=0.$$

The tension field is a smooth section in the pull-back bundle $\varphi^{-1}(TN)$. If $\varphi:(M,g)\to(N,h)$ is a Riemannian immersion, then it is a critical point of the energy functional if and only if it is a minimal immersion, i.e. a critical point of the *volume functional* (see [60]).

One can generalize harmonic maps by considering the functional obtained by integrating the squared norm of the tension field. More precisely, *biharmonic maps* are the critical points of the *bienergy functional*

$$E_2 : C^\infty(M,N)\to\mathbb{R},\quad E_2(\varphi)=\frac{1}{2}\int_M |\tau(\varphi)|^2\,v_g.$$

The associated Euler-Lagrange equation is given by the vanishing of the bitension field

$$\tau_2(\varphi)=-\Delta\tau(\varphi)-\operatorname{trace}R^N(d\varphi(\cdot),\tau(\varphi))d\varphi(\cdot)=0. \tag{0.1}$$

Obviously, harmonic maps are biharmonic. Biharmonic non-harmonic maps are called *proper-biharmonic*.

The above variational problem and the Willmore problem (see [134]) produce natural generalizations of harmonic maps and, respectively, minimal immersions. Nevertheless, biharmonic Riemannian immersions do not recover Willmore immersions, not even when the ambient space is $\mathbb{R}^n$.

The theory of biharmonic maps is an old and rich subject, initially studied due to its implications in the theory of elasticity and fluid mechanics. G.B. Airy and J.C. Maxwell were the first to study and express plane elastic problems in terms of the biharmonic equation (see [1, 94]). Later on, the theory evolved with the study of polyharmonic functions developed by E. Almansi, T. Levi-Civita, M. Nicolaescu. Biharmonic and polyharmonic functions on Riemannian manifolds were studied by R. Caddeo and L. Vanheke [28, 35], L. Sario et all (see [117]) and others.

Biharmonic maps have been extensively studied in the last decade and there are two main research directions. On the one hand, in differential geometry, a special attention has been payed to the construction of examples and classification results. Results in this direction were obtained, for example, by P. Baird [11, 12], H. Urakawa [77, 78, 128], Y.-L. Ou [110]–[113] and in [4, 14, 21, 22, 27, 29, 30, 33, 34, 42, 46, 58, 79, 127, 139].

On the other hand, from the analytic point of view, biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE and the study of their regularity is nowadays a well-developed field. Contributions in this direction were made by S.-Y.A. Chang [38], T. Lamm [84, 85], R. Moser [99, 100], P. Strzelecki [122], C. Wang [131, 132], etc.

It was proved in [61] that there exists no harmonic map from $\mathbb{T}^2$ to $\mathbb{S}^2$ (whatever the metrics chosen) in the homotopy class of Brower degree ± 1. The biharmonic maps are expected to exist where harmonic maps do not.

The interest in the theory of biharmonic maps crossed the border of differential geometry and analysis of PDE's. In computational geometry, more precisely in the field of boundary based surface design, the biharmonic Bézier surfaces are studied (see [82, 96, 97]).

The variational problem associated by considering, for a fixed map, the bienergy functional defined on the set of Riemannian metrics on the domain gave rise to the biharmonic stress-energy tensor (see [90]). This proved to be useful for obtaining new examples of proper-biharmonic maps and for the study of submanifolds with certain geometric properties, like pseudo-umbilical and parallel submanifolds.

In his studies on finite type submanifolds (see [44]), B-Y. Chen defined biharmonic Riemannian immersions, i.e. biharmonic submanifolds, in the Euclidean space as those with harmonic mean curvature vector field, that is $\Delta H = 0$, where Δ is the rough Laplacian. By considering the definition of biharmonic maps for Riemannian immersions into the Euclidean space $\mathbb{R}^n$ one recovers the notion of biharmonic submanifolds in the sense of B-Y. Chen. Although the results obtained by B-Y. Chen and his collaborators on proper-biharmonic submanifolds in Euclidean spaces are non-existence results, i.e. the only biharmonic submanifolds are the minimal ones, their techniques were adapted and led to classification results for proper-biharmonic submanifolds in Euclidean spheres where the family of such submanifolds is rather rich.

The differential geometric aspect of biharmonic submanifolds was also studied in the semi-Riemannian case (see, for example, [44, 46]).

In real space forms of nonpositive constant sectional curvature only non-existence results for proper-biharmonic submanifolds are known (see, for example, [21, 29, 43, 46, 56, 58, 75]). In the case of real space forms of positive sectional curvature the situation is completely different, and the first chapter of the present thesis concerns the classification of biharmonic submanifolds in the unit Euclidean sphere $\mathbb{S}^n$. The key ingredient is the characterization formula, obtained by splitting the bitension field in its normal and tangent components, presented in the first section. The main examples of proper-biharmonic submanifolds in $\mathbb{S}^n$, together with their immediate properties, are listed. The section ends with a partial classification result for biharmonic submanifolds with constant mean curvature (CMC) in spheres. Taking this further, in the second section we study the type of CMC proper biharmonic submanifolds in $\mathbb{S}^n$ and prove that, depending on the value of the mean curvature, they are of 1-type or of 2-type as submanifolds of $\mathbb{R}^{n+1}$. In the third section, the proper biharmonic hypersurfaces are studied from different points of view: first with respect to the number of their distinct principal curvatures, then with respect to $|A|^2$ and $|H|^2$, and, finally, the study is done with respect to the sectional, Ricci and scalar curvatures of the hypersurface. All the obtained results are rigidity results, i.e. with the imposed restrictions, the biharmonic hypersurfaces belong to the main classes of aforementioned examples. The fourth section is devoted to the study of proper-biharmonic submanifolds with parallel mean curvature vector field (PMC) in spheres, the main result of this section consisting in a partial classification. Moreover, a full classification of PMC proper-biharmonic submanifolds in spheres with parallel shape operator associated to the mean curvature vector field is presented. The chapter ends with a list of Open Problems. The results contained in this chapter can be found in [18]–[24].

Chapter 2 is devoted to the study of proper-biharmonic submanifolds in a complex space form. This subject has already been started by several authors. In [53] some pinching conditions for the second fundamental form and the Ricci curvature of a biharmonic Lagrangian submanifold of $\mathbb{C}P^n$, with parallel mean curvature vector field, were obtained. In [119], the author gave a classification of biharmonic Lagrangian surfaces of constant mean curvature in $\mathbb{C}P^2$. Then, the characterization of biharmonic constant mean curvature real hypersurfaces of $\mathbb{C}P^n$ and the classification of proper-biharmonic homogeneous real hypersurfaces of $\mathbb{C}P^n$ were obtained in [77, 78]. Our main result in Chapter 2 is a formula that relates the bitension field of a submanifold in $\mathbb{C}P^n$ and the bitension field of the associated Hopf cylinder (according to the Hopf fibration). Using this formula, many examples of proper-biharmonic submanifolds in $\mathbb{C}P^n$ were obtained. In the 2-dimensional complex projective space, by using a result of S. Maeda and T. Adachi, all proper-biharmonic curves were determined.

The Euclidean spheres proved to be a very giving environment for obtaining examples and classification results. Then, the fact that odd-dimensional spheres can be thought as a class of Sasakian space forms (which do not have constant sectional curvature, in general) led to the idea that another research direction would be the study of biharmonic submanifolds in Sasakian space forms. Following this direction, the proper-biharmonic Legendre curves and Hopf cylinders in a 3-dimensional Sasakian space form were classified in [79], whilst in [71] their parametric equations were found. In Chapter 3 we classify all proper-biharmonic Legendre curves in arbitrary dimensional Sasakian space forms, and we present a method to obtain proper-biharmonic anti-invariant submanifolds from proper-biharmonic integral submanifolds. Then, we obtain classification results for proper-biharmonic hypersurfaces. In the last part, we determine all 3-dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds in a 7-dimensional Sasakian space form and then we find these submanifolds in the unit

Euclidean 7-sphere endowed with its canonical and deformed Sasakian structures introduced by S. Tanno in [125]. We end by classifying the proper-biharmonic parallel Lagrangian submanifolds of $\mathbb{C}P^3$ by determining their horizontal lifts, with respect to the Hopf fibration, in $\mathbb{S}^7(1)$.

Some of the techniques used in the thesis are based on those developed by D. Blair, B-Y. Chen, F. Defever, M. do Carmo, J. Erbacher, J.D. Moore, K. Nomizu, P.J. Ryan, S.-T. Yau, etc.

Rezumat

Lucrarea de faţă este dedicată studiului subvarietăţilor biarmonice în forme spaţiale reale, complexe şi sasakiene. Vom prezenta, pentru început, unele idei care au incurajat studiul subvarietăţilor biarmonice şi al geometriei aplicaţiilor biarmonice şi apoi vom descrie rezultatele incluse în această teză.

Fie $C^{\infty}(M,N)$ spaţiul aplicaţiilor netede $\varphi:(M,g)\to(N,h)$ între două varietăţi riemanniene. O aplicaţie $\varphi\in C^{\infty}(M,N)$ se numeşte *armonică* dacă este un punct critic al *funcţionalei energie*

$$E: C^{\infty}(M,N)\to\mathbb{R},\quad E(\varphi)=\frac{1}{2}\int_M |d\varphi|^2\, v_g,$$

şi este caracterizată de anularea câmpului de tensiune

$$\tau(\varphi)=\operatorname{trace}\nabla d\varphi=0.$$

Campul de tensiune este o secţiune netedă în fibratul pull-back $\varphi^{-1}(TN)$.

Dacă $\varphi:(M,g)\to(N,h)$ este o imersie riemanniană, atunci este un punct critic al funcţionalei energie dacă şi numai dacă este o imersie minimală, adică un punct critic al *funcţionalei volum* (vezi [60]).

Noţiunea de aplicaţie armonică poate fi generalizată considerând funcţionala obţinută prin integrarea pătratului normei câmpului de tensiune. Mai exact, *aplicaţiile biarmonice* sunt punctele critice ale *funcţionalei bienergie*

$$E_2: C^{\infty}(M,N)\to\mathbb{R},\quad E_2(\varphi)=\frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g.$$

Ecuaţia Euler-Lagrange asociată este dată de anularea câmpului de bitensiune

$$\tau_2(\varphi)=-\Delta\tau(\varphi)-\operatorname{trace}R^N(d\varphi(\cdot),\tau(\varphi))d\varphi(\cdot)=0.$$

Evident, aplicaţiile armonice sunt biarmonice. Aplicaţiile biarmonice şi nearmonice sunt numite *biarmonice proprii.*

Problema variaţională de mai sus şi problema Willmore (vezi [134]) produc generalizări naturale ale noţiunii de aplicaţie armonică, respectiv imersie minimală. Însă imersiile riemanniene biarmonice nu sunt imersii Willmore, nici măcar în cazul în care spaţiul ambiant este $\mathbb{R}^n$.

Teoria aplicaţiilor biarmonice este un domeniu vechi şi bogat în rezultate, iniţial studiat datorită implicaţiilor sale în teoria elasticităţii şi în mecanica fluidelor. G.B. Airy şi J.C. Maxwell au fost primii care au studiat şi exprimat fenomene elastice plane in termenii ecuaţiei biarmonice (vezi [1, 94]). Mai târziu, teoria a evoluat cu studiul funcţiilor poliarmonice realizat de către E. Almansi, T. Levi-Civita, M. Nicolaescu. Funcţiile biarmonice şi poliarmonice pe varietăţi riemanniene au fost studiate de R. Caddeo şi L. Vanheke [28, 35], L. Sario et all [117] şi alţii.

Aplicaţiile biarmonice au fost intens studiate în ultimul deceniu şi există două direcţii principale de cercetare. Pe de o parte, în geometria diferenţială, o atenţie deosebită a fost acordată construcţiei de exemple şi rezultatelor de clasificare. Rezultate în această direcţie au fost obţinute, de exemplu, de P. Baird [11, 12], H. Urakawa [77, 78, 128], Y.-L. Ou [110]–[113] şi în [4, 14, 21, 22, 29, 30, 33, 34, 42, 46, 58, 79, 127, 139].

Pe de altă parte, din punct de vedere analitic, aplicaţiile biarmonice sunt soluţii ale unui sistem eliptic semi-liniar de ordin 4 de ecuaţii cu derivate parţiale, iar studiul regularităţii acestora este un domeniu de cercetare bine dezvoltat în prezent. Contribuţii în această direcţie au fost aduse de către S.-Y.A. Chang [38], T. Lamm [84, 85], R. Moser [99, 100], P. Strzelecki [122], C. Wang [131, 132], etc.

În [61] s-a demonstrat că nu există aplicaţii armonice de la $\mathbb{T}^2$ la $\mathbb{S}^2$ (indiferent de metricile alese) în clasa de omotopie de grad Brower egal cu ± 1. Se aşteaptă ca aplicaţiile biarmonice să rezolve această problemă.

Interesul manifestat pentru aplicaţiile biarmonice a depăşit graniţele geometriei diferenţiale şi ale analizei ecuaţiilor cu derivate parţiale. În geometria computaţională, mai precis în designul suprafeţelor de bord fixat, sunt intens studiate suprafeţele Bézier biarmonice (vezi [82, 96, 97]).

Problema variaţională asociată considerând, pentru o aplicaţie fixată, funcţionala bienergie definită pe mulţimea metricilor riemanniene pe domeniu a dat naştere tensorului stress-energie biarmonic (vezi [90]). Acesta s-a dovedit util în construcţia de noi exemple de aplicaţii biarmonice proprii şi în studiul subvarietăţilor cu anumite proprietăţi geometrice, cum ar fi subvarietăţile pseudo-ombelicale şi cele paralele.

În studiile sale asupra subvarietăţilor de tip finit (vezi [44]) B-Y. Chen a definit subvarietăţile biarmonice $M \subset \mathbb{R}^n$ ale spaţiului euclidian ca fiind acele subvarietăţi pentru care câmpul vectorial curbură medie este armonic, i.e. $\Delta H = 0$, unde Δ este laplaceanul pe mulţimea câmpurilor vectoriale tangente la $\mathbb{R}^n$ în lungul subvarietăţii M. Considerând definiţia aplicaţiilor biarmonice pentru imersii riemanniene în spaţiul euclidian se regăseşte noţiunea de subvarietate biarmonică în sensul lui B-Y. Chen. Notăm că toate rezultatele obţinute de către Chen şi colaboratorii săi, pentru subvarietăţi biarmonice în spaţiul euclidian, sunt rezultate de neexistenţă, adică biarmonicitatea implică minimalitate. Însă tehnicile acestora au fost adaptate şi au condus la rezultate de clasificare pentru subvarietăţi biarmonice proprii în sfere, unde familia acestor subvarietăţi este destul de bogată.

Aspectul geometric al aplicaţiilor şi subvarietăţilor biarmonice a fost tratat şi în context pseudo-riemannian (vezi, de exemplu, [44, 46]).

Toate rezultatele obţinute privitoare la subvarietăţile biarmonice proprii în forme spaţiale de curbură secţională negativă sunt de neexistenţă (vezi, de exemplu, [21, 29, 43, 46, 56, 58, 75]). În cazul formelor spaţiale de curbură secţională pozitivă situaţia se dovedeşte a fi complet diferită, iar primul capitol al prezentei teze tratează problema clasificării subvarietăţilor biarmonice proprii ale sferei euclidiene unitare $\mathbb{S}^n$. Ingredientul cheie constă în formula de caracterizare obţinută prin descompunerea câmpului de bitensiune în componentele sale, tangentă şi normală, prezentată în prima secţiune. Sunt apoi prezentate principalele exemple de subvarietăţi biarmonice proprii în $\mathbb{S}^n$, împreună cu proprietăţile lor imediate. Secţiunea se încheie cu un rezultat de clasificare parţială a subvarietăţilor biarmonice proprii de curbură medie constantă (CMC) în sfere. S-a extins acest rezultat, studiând tipul subvarietăţilor CMC biarmonice proprii în $\mathbb{S}^n$ şi s-a demonstrat că, în funcţie de valoarea curburii medii, acestea sunt fie de tip 1, fie de tip 2 ca subvarietăţi în $\mathbb{R}^{n+1}$. În a treia secţiune sunt studiate, din diferite puncte de vedere, hipersuprafeţele biarmonice proprii: mai întâi ţinând cont de numărul de curburi principale distincte, apoi în funcţie de $|A|^2$ şi $|H|^2$ şi, în final, studiul este realizat ţinând cont de curbura secţională, curbura Ricci şi curbura scalară a hipersuprafeţei. Toate rezultatele obţinute sunt rezultate de rigiditate, adică hipersuprafeţele biarmonice aparţin claselor de exemple menţionate anterior. Secţiunea a patra este dedicată studiului subvarietăţilor biarmonice proprii de câmp vectorial curbură medie paralel (PMC) în sfere, principalul rezultat constând într-o clasificare parţială. Mai mult, este prezentată clasificarea completă a subvarietăţilor PMC biarmonice proprii în sfere cu operatorul formă asociat câmpului vectorial curbură medie paralel. Capitolul se încheie cu o listă de Probleme Deschise. Rezultatele incluse în acest capitol pot fi găsite în [18]–[24].

Capitolul 2 este dedicat studiului subvarietăţilor biarmonice proprii în forme spaţiale complexe. Acest subiect a fost iniţiat de mai mulţi autori. În [53] au fost obţinute unele condiţii de pinching asupra formei a doua fundamentale şi a curburii Ricci pentru o subvarietate biarmonică lagrangiană de curbură medie paralelă în $\mathbb{C}P^n$. În [119], autorul a obţinut o clasificare a suprafeţelor lagrangiene biarmonice de curbură medie constantă în $\mathbb{C}P^2$. Apoi, în [77, 78], au fost obţinute caracterizarea hipersuprafeţelor reale biarmonice de curbură medie constantă şi clasificarea hipersuprafeţelor reale omogene biarmonice în $\mathbb{C}P^n$. Principalul nostru rezultat prezentat în Capitolul 2 este formula ce dă legătura dintre câmpul de bitensiune al unei subvarietăţi în $\mathbb{C}P^n$ şi câmpul de bitensiune al cilindrului Hopf asociat (prin intermediul fibrării Hopf). Cu ajutorul acestei formule se obţin numeroase exemple de subvarietăţi biarmonice proprii în $\mathbb{C}P^n$. Folosind un rezultat obţinut de S. Maeda şi T. Adachi, se determină toate curbele biarmonice proprii în spaţiul proiectiv complex 2-dimensional.

Sferele euclidiene s-au dovedit a fi un ambient foarte generos pentru obţinerea de exemple şi rezultate de clasificare. Mai mult, faptul că sferele de dimensiune impară pot fi privite ca o clasă de forme spaţiale sasakiene (care în general nu au curbura secţională constantă) a condus la idea că o nouă direcţie de cercetare poate fi studiul subvarietăţilor biarmonic în forme spaţiale sasakiene. Urmând această direcţie, în [79] au fost clasificate curbele Legendre şi cilindrii Hopf biarmonici proprii în forme spaţiale sasakiene 3-dimensionale, iar în [71] au fost determinate ecuaţiile parametrice ale acestora. În Capitolul 3 se clasifică toate curbele Legendre biarmonice în forme spaţiale sasakiene de dimensiune arbitrară şi se prezintă o metodă de construcţie a subvarietăţilor anti-invariante biarmonice proprii pornind de la subvarietăţi integrale biarmonice proprii. Se obţin apoi rezultate de clasificare pentru hipersuprafeţe biarmonice proprii. În ultima parte sunt determinate toate subvarietăţile integrale $\mathcal{C}$-paralele, 3-dimensionale, biarmonice proprii ale unei forme spaţiale sasakiene 7-dimensionale şi apoi sunt obţinute aceste subvarietăţi în sfera unitate 7-dimensională înzestrată cu structura sasakiană canonică şi cu structurile sasakiene deformate introduse de S. Tanno în [125]. În încheiere se prezintă clasificarea subvarietăţilor lagrangiene paralele biarmonice proprii în $\mathbb{C}P^3$ prin determinarea lifturilor orizontale, în raport cu fibrarea Hopf, în $\mathbb{S}^7(1)$.

Tehnicile folosite în această teză sunt bazate pe tehnici dezvoltate de D. Blair, B-Y. Chen, F. Defever, M. do Carmo, J. Erbacher, J.D. Moore, K. Nomizu, P.J. Ryan, S.-T. Yau, etc.

Contents

Generalities

Conventions

Throughout this work all manifolds, metrics, maps are assumed to be smooth, i.e. in the C^∞ category. All manifolds are assumed to be connected.

The following sign convention is used for the curvature tensor field of a Riemannian manifold (M, g)

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

where $X, Y \in C(TM)$ are vector fields on M and ∇ is the Levi-Civita connection of the manifold. Moreover, the Ricci tensor field Ricci and the scalar curvature s are defined by

$$\langle \mathrm{Ricci}(X), Y \rangle = \mathrm{Ricci}(X, Y) = \mathrm{trace}(Z \to R(Z, X)Y)), \quad s = \mathrm{trace}\,\mathrm{Ricci},$$

where $X, Y, Z \in C(TN)$.

For a map $\varphi : M \to N$ between two Riemannian manifolds, the rough Laplacian on the pull-back bundle $\varphi^{-1}(TN)$ is defined by

$$\Delta^\varphi V = -\,\mathrm{trace}(\nabla^\varphi)^2 V,$$

where $V \in C(\varphi^{-1}(TN))$ is a smooth section. Here ∇^φ denotes the connection of $\varphi^{-1}(TN)$ defined by the Levi-Civita connection of (N, h). When no confusion can occur, we shall denote $\Delta^\varphi V$ by ΔV and $\nabla^\varphi V$ by ∇V.

Submanifolds in Riemannian manifolds

In order to fix the notations, we recall here only the fundamental equations of first order for a submanifold in a Riemannian manifold. These equations define the second fundamental form, the shape operator and the connection in the normal bundle.

Let $\varphi : (M, g) \to (N, h)$ be a Riemannian immersion. For each $p \in M$, $T_{\varphi(p)}N$ splits as an orthogonal direct sum

$$T_{\varphi(p)}N = d\varphi(T_pM) \oplus d\varphi(T_pM)^\perp, \tag{0.2}$$

and $NM = \bigcup_{p \in M} d\varphi(T_pM)^\perp$ is referred to as the normal bundle of φ, or of M in N.

Denote by ∇ and ∇^N the Levi-Civita connections on M and N, respectively, and by ∇^φ the induced connection in the pull-back bundle $\varphi^{-1}(TN) = \bigcup_{p\in M} T_{\varphi(p)}N$. Taking into account the decomposition in (0.2), one has

$$\nabla^\varphi_X d\varphi(Y) = d\varphi(\nabla_X Y) + B(X,Y), \qquad \forall\, X, Y \in C(TM),$$

where $B \in C(\odot^2 T^*M \otimes NM)$ is called the second fundamental form of M in N. Here T^*M denotes the cotangent bundle of M. The mean curvature vector field of M in N is defined by $H = (\operatorname{trace} B)/m \in C(NM)$ and the mean curvature function of M is $|H|$.

Furthermore, if $\eta \in C(NM)$, then

$$\nabla^\varphi_X \eta = -d\varphi(A_\eta(X)) + \nabla^\perp_X \eta, \qquad \forall\, X \in C(TM),$$

where $A_\eta \in C(T^*M \otimes TM)$ is called the shape operator of M in N in the direction of η, and $\nabla^\perp$ is a connection on sections of NM, called the induced connection in the normal bundle. Moreover, $\langle B(X,Y), \eta\rangle = \langle A_\eta(X), Y\rangle$, for all $X, Y \in C(TM)$, $\eta \in C(NM)$.

When confusion is unlikely, locally, we identify M with its image, X with $d\varphi(X)$ and we replace $\nabla^\varphi_X d\varphi(Y)$ with $\nabla^N_X Y$. With these identifications in mind, we write

$$\nabla^N_X Y = \nabla_X Y + B(X,Y),$$

and

$$\nabla^N_X \eta = -A_\eta(X) + \nabla^\perp_X \eta.$$

We shall assume that the Gauss, Codazzi and Ricci equations are known.

Chapter 1

Classification results for biharmonic submanifolds in $\mathbb{S}^n$

1.1 Introduction

Let $\varphi : M \to (N, h)$ be a Riemannian immersion of a manifold M into a Riemannian manifold (N, h). We say that φ is *biharmonic*, or M is a *biharmonic submanifold*, if its mean curvature vector field H satisfies the following equation

$$\tau_2(\varphi) = -m \left(\Delta H + \operatorname{trace} R^N(d\varphi(\cdot), H)d\varphi(\cdot)\right) = 0, \tag{1.1}$$

where Δ denotes the rough Laplacian on sections of the pull-back bundle $\varphi^{-1}(TN)$ and R^N denotes the curvature operator on (N, h).

Obviously, any minimal immersion, i.e. $H = 0$, is biharmonic. The non-harmonic biharmonic immersions are called *proper-biharmonic*.

The study of proper-biharmonic submanifolds is nowadays becoming a very active subject and its popularity initiated with the challenging conjecture of B-Y. Chen (see the recent book [39]): *any biharmonic submanifold in the Euclidean space is minimal.*

Chen's conjecture was generalized to: *any biharmonic submanifold in a Riemannian manifold with nonpositive sectional curvature is minimal*, but this was proved not to be true. Indeed, in [113], Y.-L. Ou and L. Tang constructed examples of proper-biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature.

Yet, the conjecture is still open in its full generality for ambient spaces with constant nonpositive sectional curvature, although it was proved to be true in numerous cases when additional geometric properties for the submanifolds were assumed (see, for example, [21, 29, 43, 56, 58, 75]).

By way of contrast there are several families of examples of proper-biharmonic submanifolds in the n-dimensional unit Euclidean sphere $\mathbb{S}^n$. For simplicity we shall denote these classes by **B1**, **B2**, **B3** and **B4**.

The goal of this chapter is to present the results obtained until now for proper-biharmonic submanifolds in $\mathbb{S}^n$. The main purpose, which we are working for, is to obtain the *complete classification of proper-biharmonic submanifolds in* $\mathbb{S}^n$. This program was initiated for the very first time in [80] and then developed in [17] – [24], [29, 30, 102, 103, 109].

At the beginning of the chapter, two important properties of proper-biharmonic submanifolds in $\mathbb{S}^n$ are presented: if the mean curvature $|H|$ of such a submanifold is constant, then it is bounded, $|H| \in (0,1]$; and the submanifold, now as a submanifold of the ambient space $\mathbb{R}^{n+1}$, is of 1-type if $|H| = 1$, or of 2-type if $|H| \in (0,1)$.

In this chapter, by a rigidity result for proper-biharmonic submanifolds we mean: *find under what conditions a proper-biharmonic submanifold in $\mathbb{S}^n$ is one of the main examples* **B1**, **B2**, **B3** *and* **B4**.

We prove rigidity results for the following types of submanifolds in $\mathbb{S}^n$: hypersurfaces with at most two distinct principal curvatures everywhere, constant mean curvature (CMC) compact hypersurfaces with three distinct principal curvatures everywhere, Dupin hypersurfaces; hypersurfaces, both compact and non-compact, with bounded norm of the second fundamental form; hypersurfaces satisfying intrinsic geometric properties; parallel mean curvature vector field (PMC) submanifolds; parallel submanifolds. In the study of complete non-compact proper-biharmonic hypersurfaces with bounded norm of the second fundamental form we used the Omori-Yau Maximum Principle.

We note that, for compact proper-biharmonic hypersurfaces with bounded norm of the second fundamental form an interesting connection can be made with the case of minimal hypersurfaces with the same property.

Moreover, we include in this chapter two results of J.H. Chen published in [48], in Chinese. We give a complete proof of these results using the invariant formalism and shortening the original proofs.

1.2 Biharmonic submanifolds in $\mathbb{S}^n$

The key ingredient in the study of biharmonic submanifolds is the splitting of the bitension field with respect to its normal and tangent components. In the case when the ambient space is the unit Euclidean sphere we have the following characterization.

Theorem 1.1 ([44, 109]). *An immersion $\varphi : M^m \to \mathbb{S}^n$ is biharmonic if and only if*

$$\begin{cases} \Delta^{\perp} H + \operatorname{trace} B(\cdot, A_H \cdot) - m\, H = 0, \\ 2 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + \dfrac{m}{2} \operatorname{grad} |H|^2 = 0, \end{cases} \tag{1.2}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $|H|$ the mean curvature function, $\nabla^{\perp}$ and $\Delta^{\perp}$ the connection and the Laplacian in the normal bundle of φ, respectively.

Proof. From (1.1), the map φ is biharmonic if and only if

$$\Delta H - mH = 0. \tag{1.3}$$

Consider now $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on M, geodesic at $p \in M$.

With the usual local identification of M with $\varphi(M)$, at p we have

$$\begin{aligned}
\Delta H &= -\sum_{i=1}^m \nabla^{\mathbb{S}^n}_{E_i}\nabla^{\mathbb{S}^n}_{E_i} H = -\sum_{i=1}^m \{\nabla^{\mathbb{S}^n}_{E_i}(\nabla^{\perp}_{E_i} H - A_H(E_i))\} \\
&= -\sum_{i=1}^m \{\nabla^{\perp}_{E_i}\nabla^{\perp}_{E_i} H - A_{\nabla^{\perp}_{E_i}H}(E_i) - \nabla_{E_i} A_H(E_i) - B(E_i, A_H(E_i))\} \\
&= \Delta^{\perp} H + \operatorname{trace} B(\cdot, A_H \cdot) + \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) + \operatorname{trace} \nabla A_H.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^m \nabla_{E_i} A_H(E_i) &= \sum_{i,j} \langle \nabla_{E_i} A_H(E_i), E_j\rangle E_j = \sum_{i,j} E_i \langle A_H(E_i), E_j\rangle E_j \\
&= \sum_{i,j} E_i \langle B(E_i, E_j), H\rangle E_j = \sum_{i,j} E_i \langle \nabla^{\mathbb{S}^n}_{E_j} E_i, H\rangle E_j \\
&= \sum_{i,j} \big\{\langle \nabla^{\mathbb{S}^n}_{E_i}\nabla^{\mathbb{S}^n}_{E_j} E_i, H\rangle + \langle \nabla^{\mathbb{S}^n}_{E_j} E_i, \nabla^{\mathbb{S}^n}_{E_i} H\rangle\big\} E_j \\
&= \sum_{i,j} \big\{\langle \nabla^{\mathbb{S}^n}_{E_i}\nabla^{\mathbb{S}^n}_{E_j} E_i, H\rangle + \langle B(E_i, E_j), \nabla^{\perp}_{E_i} H\rangle\big\} E_j \\
&= \sum_{i,j} \langle \nabla^{\mathbb{S}^n}_{E_i}\nabla^{\mathbb{S}^n}_{E_j} E_i, H\rangle + \sum_i A_{\nabla^{\perp}_{E_i}H}(E_i)
\end{aligned}$$

and, since at p,

$$\begin{aligned}
\sum_{i=1}^m \langle \nabla^{\mathbb{S}^n}_{E_i}\nabla^{\mathbb{S}^n}_{E_j} E_i, H\rangle &= \sum_{i=1}^m \langle R^{\mathbb{S}^n}(E_i, E_j)E_i + \nabla^{\mathbb{S}^n}_{E_j}\nabla^{\mathbb{S}^n}_{E_i} E_i + \nabla^{\mathbb{S}^n}_{[E_i,E_j]} E_i, H\rangle \\
&= \langle -mcE_j, H\rangle + \sum_{i=1}^m \langle \nabla^{\mathbb{S}^n}_{E_j} B(E_i, E_i), H\rangle = m\langle \nabla^{\mathbb{S}^n}_{E_j} H, H\rangle \\
&= \frac{m}{2} E_j(|H|^2),
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{i=1}^m \{A_{\nabla^{\perp}_{E_i}H}(X_i) + \nabla_{E_i} A_H(E_i)\} &= 2\sum_{i=1}^m A_{\nabla^{\perp}_{E_i}H}(E_i) + \frac{m}{2}(d|H|^2)^{\sharp} \\
&= 2 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) + \frac{m}{2} \operatorname{grad}(|H|^2).
\end{aligned}$$

Thus, by replacing the expression for ΔH in (1.3) we obtain that φ is biharmonic if and only if

$$-\Delta^{\perp} H - \operatorname{trace} B(\cdot, A_H \cdot) + mcH = 2 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) + \frac{m}{2} \operatorname{grad}(|H|^2). \tag{1.4}$$

Since the left hand side of (1.4) is normal, and the right hand side is tangent to M, we conclude. $\square$

In the codimension one case, denoting by $A = A_\eta$ the shape operator with respect to a (local) unit section η in the normal bundle and putting $f = (\operatorname{trace} A)/m$, the above result reduces to the following.

Corollary 1.2 ([109]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be an orientable hypersurface. Then φ is biharmonic if and only if*

$$\begin{cases} \text{(i)} \quad \Delta f = (m - |A|^2) f, \\ \\ \text{(ii)} \quad A(\operatorname{grad} f) = -\dfrac{m}{2} f \operatorname{grad} f. \end{cases} \tag{1.5}$$

A special class of immersions in $\mathbb{S}^n$ consists of the parallel mean curvature immersions (PMC), that is immersions such that $\nabla^\perp H = 0$. For this class of immersions Theorem 1.1 reads as follows.

Corollary 1.3 ([23]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC immersion. Then φ is biharmonic if and only if*

$$\operatorname{trace} B(A_H(\cdot), \cdot) = mH, \tag{1.6}$$

or equivalently,

$$\begin{cases} \langle A_H, A_\xi \rangle = 0, \quad \forall \xi \in C(NM) \text{ with } \xi \perp H, \\ \\ |A_H|^2 = m|H|^2, \end{cases} \tag{1.7}$$

where NM denotes the normal bundle of M in $\mathbb{S}^n$.

We now list the main examples of proper-biharmonic immersions in $\mathbb{S}^n$.

B1. The canonical inclusion of the small hypersphere

$$\mathbb{S}^{n-1}(1/\sqrt{2}) = \Big\{ (x, 1/\sqrt{2}) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, |x|^2 = 1/2 \Big\} \subset \mathbb{S}^n. \tag{1.8}$$

B2. The canonical inclusion of the standard (extrinsic) products of spheres

$$\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) = \big\{ (x, y) \in \mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2+1}, |x|^2 = |y|^2 = 1/2 \big\} \subset \mathbb{S}^n, \tag{1.9}$$

$n_1 + n_2 = n - 1$ and $n_1 \neq n_2$.

B3. The maps $\varphi = \imath \circ \phi : M \to \mathbb{S}^n$, where $\phi : M \to \mathbb{S}^{n-1}(1/\sqrt{2})$ is a minimal immersion, and $\imath : \mathbb{S}^{n-1}(1/\sqrt{2}) \to \mathbb{S}^n$ denotes the canonical inclusion.

B4. The maps $\varphi = \imath \circ (\phi_1 \times \phi_2) : M_1 \times M_2 \to \mathbb{S}^n$, where $\phi_i : M_i^{m_i} \to \mathbb{S}^{n_i}(1/\sqrt{2})$, $0 < m_i \leq n_i$, $i = 1, 2$, are minimal immersions, $m_1 \neq m_2$, $n_1 + n_2 = n - 1$, and $\imath : \mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) \to \mathbb{S}^n$ denotes the canonical inclusion.

Remark 1.4. (i) The proper-biharmonic immersions of class **B3** are pseudo-umbilical, i.e. $A_H = |H|^2 \operatorname{Id}$, have parallel mean curvature vector field and mean curvature $|H| = 1$. Clearly, $\nabla A_H = 0$.

(ii) The proper-biharmonic immersions of class **B4** are no longer pseudo-umbilical, but still have parallel mean curvature vector field and their mean curvature is $|H| = |m_1 - m_2|/m \in (0, 1)$, where $m = m_1 + m_2$. Moreover, $\nabla A_H = 0$ and the principal curvatures in the direction of H, i.e. the eigenvalues of A_H, are constant on M and given by $\lambda_1 = \ldots = \lambda_{m_1} = (m_1 - m_2)/m$, $\lambda_{m_1+1} = \ldots = \lambda_{m_1+m_2} = -(m_1 - m_2)/m$. Specific B4 examples were given by W. Zhang in [139] and generalized in [20, 133].

Example **B2** was found in [80], while Example **B1** was derived in [30]. The two families of examples described in Example **B3** and Example **B4** were constructed in [29]. Moreover, Example **B3** is a consequence of the following property.

Theorem 1.5 ([29]). *Let $\psi : M \to \mathbb{S}^{n-1}(a)$ be a minimal submanifold in a small hypersphere $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$, of radius $a \in (0, 1)$, and denote by $\imath : \mathbb{S}^{n-1}(a) \to \mathbb{S}^n$ the inclusion map. Then $\varphi = \imath \circ \psi : M \to \mathbb{S}^n$ is proper-biharmonic if and only if $a = 1/\sqrt{2}$.*

Example **B4** is a consequence of the following result.

Theorem 1.6 ([29]). *Let $\psi_1 : M_1^{m_1} \to \mathbb{S}^{n_1}(a)$ and $\psi_2 : M_2^{m_2} \to \mathbb{S}^{n_2}(b)$ be two minimal submanifolds, where $n_1 + n_2 = n - 1$, $a^2 + b^2 = 1$, and denote by $\imath : \mathbb{S}^{n_1}(a) \times \mathbb{S}^{n_2}(b) \to \mathbb{S}^n$ the inclusion map. Then $\varphi = \imath \circ (\psi_1 \times \psi_2) : M_1 \times M_2 \to \mathbb{S}^n$ is proper-biharmonic if and only if $a = b = 1/\sqrt{2}$ and $m_1 \neq m_2$.*

When a biharmonic immersion has constant mean curvature (CMC) the following bound for $|H|$ holds.

Theorem 1.7 ([108]). *Let $\varphi : M \to \mathbb{S}^n$ be a CMC proper-biharmonic immersion. Then $|H| \in (0, 1]$, and $|H| = 1$ if and only if φ induces a minimal immersion of M into $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$, that is φ is* **B3**.

Proof. Let M be a CMC biharmonic submanifold of $\mathbb{S}^n$. The first equation of (1.2) implies that

$$\langle \Delta^\perp H, H \rangle = m|H|^2 - |A_H|^2,$$

and by using the Weitzenböck formula,

$$\frac{1}{2}\Delta |H|^2 = \langle \Delta^\perp H, H \rangle - |\nabla^\perp H|^2,$$

we obtain

$$m|H|^2 = |A_H|^2 + |\nabla^\perp H|^2. \tag{1.10}$$

Let now $\{X_i\}$ be a local orthonormal basis such that $A_H(X_i) = \lambda_i X_i$. From

$$\lambda_i = \langle A_H(X_i), X_i \rangle = \langle B(X_i, X_i), H \rangle$$

and

$$\sum \lambda_i = m|H|^2, \qquad \sum (\lambda_i)^2 = |A_H|^2,$$

using (1.10) we obtain

$$\sum \lambda_i = \sum (\lambda_i)^2 + |\nabla^\perp H|^2 \geq \frac{(\sum \lambda_i)^2}{m} + |\nabla^\perp H|^2. \tag{1.11}$$

Thus

$$m|H|^2 \geq m|H|^4 + |\nabla^\perp H|^2.$$

Consequently, if $|H| > 1$, the last inequality leads to a contradiction.

If $|H| = 1$, then the last inequality implies $\nabla^\perp H = 0$ and $\sum(\lambda_i)^2 = \dfrac{(\sum \lambda_i)^2}{m} = m$, thus we get $\lambda_1 = \ldots = \lambda_m$. Therefore M is PMC and pseudo-umbilical in $\mathbb{S}^n$. This implies that M is a minimal submanifold of a hypersphere $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$ (see, for example, [45]), and from Theorem 1.5 we conclude. □

1.3 On the type of biharmonic submanifolds in $\mathbb{S}^n$

Definition 1.8 ([42, 44])**.** A submanifold $\phi : M \to \mathbb{R}^{n+1}$ is called of *finite type* if it can be expressed as a finite sum of $\mathbb{R}^{n+1}$-valued eigenmaps of the Laplacian Δ of M, i.e.

$$\phi = \phi_0 + \phi_{t_1} + \ldots + \phi_{t_k}, \tag{1.12}$$

where $\phi_0 \in \mathbb{R}^{n+1}$ is a constant vector, $\phi_{t_i} : M \to \mathbb{R}^{n+1}$ are non-constant maps satisfying $\Delta\phi_{t_i} = \lambda_{t_i}\phi_{t_i}$, $i = 1, \ldots, k$. If, in particular, all eigenvalues λ_{t_i} are assumed to be mutually distinct, the submanifold is said to be of *k-type* and (1.12) is called the *spectral decomposition* of ϕ.

Remark 1.9. If M is compact the immersion $\phi : M \to \mathbb{R}^{n+1}$ admits a unique spectral decomposition $\phi = \phi_0 + \sum_{i=1}^{\infty} \phi_i$, where ϕ_0 is the *center of mass*. Then, it is of k-type if only k terms of $\{\phi_i\}_{i=1}^{\infty}$ are not vanishing. In the non-compact case the spectral decomposition $\phi = \phi_0 + \sum_{i=1}^{\infty} \phi_i$ is not guaranteed. Nonetheless, if Definition 1.8 is satisfied, the spectral decomposition is unique. Notice also that, in the non-compact case, the harmonic component of the spectral decomposition is not necessarily constant. Finite type submanifolds with non-constant harmonic component are called *null finite type* submanifolds.

The following result provides us a necessary and a sufficient condition for a submanifold to be of finite type.

Theorem 1.10 ([44, 47])**.** *Let $\phi : M \to \mathbb{R}^{n+1}$ be a Riemannian immersion.*

(i) If M is of finite k-type, there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and a monic polynomial with simple roots P of degree k with $P(\Delta)(\phi - \phi_0) = 0$.

(ii) If there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and a polynomial P with simple roots such that $P(\Delta)(\phi - \phi_0) = 0$, then M is of finite k-type with $k \leq \text{degree}(P)$.

The following version shall also be used.

Theorem 1.11 ([44, 47])**.** *Let $\phi : M \to \mathbb{R}^{n+1}$ be a Riemannian immersion.*

(i) If M is of finite k-type, there exists a monic polynomial P of degree $k-1$ or k with $P(\Delta)H^0 = 0$.

(ii) If there exists a polynomial P with simple roots such that $P(\Delta)H^0 = 0$, then M is of infinite type or of finite k-type with $k - 1 \leq \text{degree}(P)$.

Here H^0 *denotes the mean curvature vector field of* M *in* $\mathbb{R}^{n+1}$.

A well known result of T. Takahashi can be rewritten as the classification of 1-type submanifolds in $\mathbb{R}^{n+1}$.

Theorem 1.12 ([124]). *A submanifold* $\phi : M \to \mathbb{R}^{n+1}$ *is of* 1*-type if and only if either* ϕ *is a minimal immersion in* $\mathbb{R}^{n+1}$, *or* ϕ *induces a minimal immersion of* M *in a hypersphere of* $\mathbb{R}^{n+1}$.

Definition 1.13. A submanifold $\varphi : M \to \mathbb{S}^n$ is said to be of *finite type* if it is of finite type as a submanifold of $\mathbb{R}^{n+1}$, where $\mathbb{S}^n$ is canonically embedded in $\mathbb{R}^{n+1}$. Moreover, a non-null finite type submanifold in $\mathbb{S}^n$ is said to be *mass-symmetric* if the constant vector ϕ_0 of its spectral decomposition is the center of the hypersphere $\mathbb{S}^n$, i.e. $\phi_0 = 0$.

Remark 1.14. By Theorem 1.12, biharmonic submanifolds of class B3 are 1-type submanifolds. Indeed, the immersion $\phi : M \to \mathbb{R}^{n+1}$ of M in $\mathbb{R}^{n+1}$ has the spectral decomposition

$$\phi = \phi_0 + \phi_p,$$

where $\phi_0 = (0, 1/\sqrt{2})$, $\phi_p : M \to \mathbb{R}^{n+1}$, $\phi_p(x) = (\psi(x), 0)$ and $\Delta\phi_p = 2m\phi_p$.

Moreover, biharmonic submanifolds of class B4 are mass-symmetric 2-type submanifolds. Indeed, $\phi : M_1 \times M_2 \to \mathbb{R}^{n+1}$ has the spectral decomposition

$$\phi = \phi_p + \phi_q,$$

where $\phi_p(x, y) = (\psi_1(x), 0)$, $\phi_q(x, y) = (0, \psi_2(y))$, $\Delta\phi_p = 2m_1\phi_p$, $\Delta\phi_q = 2m_2\phi_q$.

Let $\varphi : M \to \mathbb{S}^n$ be a submanifold in $\mathbb{S}^n$ and denote by $\phi = \mathbf{i} \circ \varphi : M \to \mathbb{R}^{n+1}$ the immersion of M in $\mathbb{R}^{n+1}$. Denote by H the mean curvature vector field of M in $\mathbb{S}^n$ and by H^0 the mean curvature vector field of M in $\mathbb{R}^{n+1}$.

The mean curvature vector fields H^0 and H are related by $H^0 = H - \phi$. Moreover, we have

$$\langle H, \phi \rangle = 0, \quad \langle H^0, H \rangle = |H|^2, \quad \langle H^0, \phi \rangle = -1. \tag{1.13}$$

Following [29], the bitension field of φ can be written as

$$\tau_2(\varphi) = -m\Delta H^0 + 2m^2 H^0 + m^2\{2 - |H^0|^2\}\phi.$$

Thus, $\tau_2(\varphi) = 0$ if and only if

$$\Delta H^0 - 2mH^0 + m(|H|^2 - 1)\phi = 0. \tag{1.14a}$$

or equivalently, since $\Delta\phi = -mH^0$,

$$\Delta^2\phi - 2m\Delta\phi - m^2(|H|^2 - 1)\phi = 0, \tag{1.14b}$$

In [21, Theorem 3.1] we proved that CMC compact proper biharmonic submanifolds in $\mathbb{S}^n$ are of 1-type or 2-type. This result can be generalized to the following.

Theorem 1.15 ([17]). *Let* $\varphi : M \to \mathbb{S}^n$ *be a proper-biharmonic submanifold, not necessarily compact, in the unit Euclidean sphere* $\mathbb{S}^n$. *Denote by* $\phi = \mathbf{i} \circ \varphi : M \to \mathbb{R}^{n+1}$ *the immersion of* M *in* $\mathbb{R}^{n+1}$, *where* $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ *is the canonical inclusion map. Then* *(i) M is a 1-type submanifold if and only if $|H| = 1$. In this case, $\phi = \phi_0 + \phi_p$, $\Delta\phi_p = 2m\phi_p$, and $\phi_0 \in \mathbb{R}^{n+1}$, $|\phi_0| = 1/\sqrt{2}$.*

(ii) M is a 2-type submanifold if and only if $|H| =$ constant, $|H| \in (0, 1)$. In this case, $\phi = \phi_p + \phi_q$, $\Delta\phi_p = m(1 - |H|)\phi_p$, $\Delta\phi_q = m(1 + |H|)\phi_q$.

Proof. In order to prove (i), notice that the converse is obvious, by Theorem 1.12 and Theorem 1.7.

Let us suppose that M is a 1-type submanifold. From Theorem 1.11(i) follows that there exists $a \in \mathbb{R}$ such that
$$\Delta H^0 = aH^0. \tag{1.15}$$
Equations (1.14a) and (1.15) imply
$$(2m - a)H^0 - m(|H|^2 - 1)\phi = 0,$$
and by considering the scalar product with H and using (1.13), since M is proper-biharmonic, we get $a = 2m$ and
$$m(|H|^2 - 1)\phi = 0.$$
Thus $|H| = 1$. Now, as the map ϕ can not be harmonic, (1.14b) leads to the spectral decomposition $\phi = \phi_0 + \phi_p$, $\Delta\phi_p = 2m\phi_p$. Since $\Delta\phi = -mH^0$, taking into account the relation between H and H^0, we obtain $2\phi_0 = \phi + H$. Since $|\phi| = 1 = |H|$, and H is orthogonal to ϕ, we conclude that $|\phi_0| = 1/\sqrt{2}$.

Let us now prove (ii). The converse of (ii) follows immediately. Indeed, from (1.14b), if $|H| =$ constant, $|H| \in (0, 1)$, then choosing the constant vector $\phi_0 = 0$ and the polynomial with simple roots
$$P(\Delta) = \Delta^2 - 2m\Delta^1 - m^2(|H|^2 - 1)\Delta^0,$$
we are in the hypotheses of Theorem 1.10(ii). Thus M is of finite k-type, with $k \leq 2$. Taking into account (i), since $|H| \in (0, 1)$, this implies that M is a 2-type submanifold with
$$\phi = \phi_p + \phi_q,$$
with corresponding eigenvalues $\lambda_p = m(1 - |H|)$, $\lambda_q = m(1 + |H|)$. Also, notice that
$$\phi_p = \frac{\lambda_q}{\lambda_q - \lambda_p}\phi - \frac{1}{\lambda_q - \lambda_q}\Delta\phi, \qquad \phi_q = -\frac{\lambda_p}{\lambda_q - \lambda_p}\phi + \frac{1}{\lambda_q - \lambda_q}\Delta\phi,$$
which are smooth non-zero maps.

Suppose now that M is a 2-type submanifold. From Theorem 1.10(i) follows that there exist a constant vector $\phi_0 \in \mathbb{R}^{n+1}$ and $a, b \in \mathbb{R}$ such that
$$\Delta H^0 = aH^0 + b(\phi - \phi_0). \tag{1.16}$$
Equations (1.14a) and (1.16) lead to
$$(2m - a)H^0 - (m(|H|^2 - 1) + b)\phi + b\phi_0 = 0. \tag{1.17}$$

We have to consider two cases.
Case 1. If $b = 0$, i.e. M is a null 2-type submanifold, by taking the scalar product with H in (1.17) and using (1.13), since M is proper biharmonic, we get $a = 2m$ and $|H| = 1$. By (i), this leads to a contradiction.
Case 2. If $b \neq 0$, we shall prove that $\operatorname{grad}|H|^2 = 0$ on M, and therefore $|H|$ is constant on M. Indeed, locally, by taking the scalar product with $X \in C(TU)$ in (1.17), we obtain $\langle \phi_0, X\rangle = 0$, for all $X \in C(TU)$, i.e. the component of ϕ_0 tangent to U vanishes

$$(\phi_0)^\top = 0, \tag{1.18}$$

where U denotes an arbitrarily open set in M. Take now the scalar product with ϕ in (1.17) and use (1.13). We obtain

$$-2m + a - m(|H|^2 - 1) - b + b\langle \phi_0, \phi\rangle = 0,$$

and, by differentiating,

$$m \operatorname{grad}|H|^2 = b \operatorname{grad}\langle \phi_0, \phi\rangle. \tag{1.19}$$

Now, by considering $\{E_i\}_{i=1}^m$ to be a local orthonormal frame field on U, we have

$$\begin{aligned}\operatorname{grad}\langle \phi_0, \phi\rangle &= \sum_{i=1}^m E_i(\langle \phi_0, \phi\rangle)E_i = \sum_{i=1}^m \langle \phi_0, \nabla^0_{E_i}\phi\rangle E_i = \sum_{i=1}^m \langle \phi_0, E_i\rangle E_i \\ &= (\phi_0)^\top. \end{aligned} \tag{1.20}$$

This, together with equations (1.18) and (1.19), leads to $\operatorname{grad}|H|^2 = 0$ on U.

Now, as $|H|$ is constant on M, using Theorem 1.7, we conclude the proof. □

Remark 1.16. If M is biharmonic of 1-type, then we can prove, in a more geometric manner, that φ is B3 (see [15]). Indeed, if ϕ is a 1-type Riemannian immersion of eigenvalue $2m$ and $|\phi_0| = 1/\sqrt{2}$, then

$$\phi = \phi_0 + \phi_p, \qquad \Delta\phi_p = 2m\phi_p.$$

As $\phi_p : M \to \mathbb{R}^{n+1}$ is also a Riemannian immersion, from a result of T. Takahashi (see [124]), we have that $\phi_p(M)$ is contained in the hypersphere $\mathbb{S}^n(1/\sqrt{2})$ of $\mathbb{R}^{n+1}$ (centered at the origin). Moreover, ϕ_p, thought of as a map into $\mathbb{S}^n(1/\sqrt{2})$, is minimal. From here, $\phi(M)$ is contained in the hypersphere $\mathbb{S}^n_{\phi_0}(1/\sqrt{2})$ centered at ϕ_0 and ϕ, thought of as a map into $\mathbb{S}^n_{\phi_0}(1/\sqrt{2})$, is minimal. Since $|\phi_0| = |\phi_p| = 1/\sqrt{2}$, we get that $\phi(M)$ lies at the intersection between $\mathbb{S}^n_{\phi_0}(1/\sqrt{2})$ and the hyperplane $\langle x - \phi_0, \phi_0\rangle = 0$, thus $\phi(M) \subset \mathbb{S}^{n-1}_{\phi_0}(1/\sqrt{2})$. Next, since the inclusion $\mathbb{S}^{n-1}_{\phi_0}(1/\sqrt{2}) \to \mathbb{S}^n_{\phi_0}(1/\sqrt{2})$ is totally geodesic, we have that ϕ, as a map into $\mathbb{S}^{n-1}_{\phi_0}(1/\sqrt{2})$, is minimal. This last minimal map is the desired map ψ, where $\mathbb{S}^{n-1}_{\phi_0}(1/\sqrt{2})$ is identified with $\mathbb{S}^{n-1}(1/\sqrt{2})$.

1.4 Biharmonic hypersurfaces in spheres

The first case to look at is that of CMC proper-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$.

Theorem 1.17 ([21]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a CMC proper-biharmonic hypersurface. Then*

(i) $|A|^2 = m$;

(ii) the scalar curvature s is constant and positive, $s = m^2(1 + |H|^2) - 2m$.

Proof. We obtain (i) as an immediate consequence of (1.5).

For (ii), from the Gauss equation we obtain

$$\operatorname{Ricci}(X, Y) = (m-1)\langle X, Y\rangle + \langle A(X), Y\rangle \operatorname{trace} A - \langle A(X), A(Y)\rangle.$$

Since $|A|^2 = m$, by considering the trace, we conclude. □

Remark 1.18. In the minimal case the condition $|A|^2 = m$ is exhaustive. In fact a minimal hypersurface in $\mathbb{S}^{m+1}$ with $|A|^2 = m$ is a minimal standard product of spheres (see [52, 87]). We point out that the full classification of CMC hypersurfaces in $\mathbb{S}^{m+1}$ with $|A|^2 = m$, therefore biharmonic, is not known.

As a direct consequence of [105, Theorem 2] we have the following result.

Theorem 1.19 ([16]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a CMC proper-biharmonic hypersurface. Assume that M has non-negative sectional curvature. Then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

In the following we shall no longer assume that the biharmonic hypersurfaces have constant mean curvature, and we shall split our study in three cases. In Case 1 we shall study the proper-biharmonic hypersurfaces with respect to the number of their distinct principal curvatures, in Case 2 we shall study them with respect to $|A|^2$ and $|H|^2$, and in Case 3 the study will be done with respect to the sectional, Ricci and scalar curvatures of the hypersurface.

1.4.1 Case 1

Obviously, if $\varphi : M^m \to \mathbb{S}^{m+1}$ is an umbilical proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$, then $\varphi(M)$ is an open part of $\mathbb{S}^m(1/\sqrt{2})$.

When the hypersurface has at most two or exactly three distinct principal curvatures everywhere we obtain the following rigidity results.

Theorem 1.20 ([21]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a hypersurface. Assume that φ is proper-biharmonic with at most two distinct principal curvatures everywhere. Then φ is CMC.*

Proof. Suppose that φ is not CMC. Then, there exists an open subset U of M such that $f = |H| > 0$, $\operatorname{grad} f \neq 0$ at any point of U; $\eta = H/|H|$. Note that U can not be made out only of umbilical points (otherwise it would be CMC). We can then assume that there exists a point $q \in U$ which is not umbilical. Then, eventually by restricting U, we can assume that $A \neq f \operatorname{Id}$ at every point of U, thus A has exactly two distinct principal curvatures on U. Recall that, as A has exactly two distinct principal curvatures, the multiplicities of its principal curvatures are constant and the principal curvatures are smooth (see [116] or [13]). Thus A is diagonalizable with respect to a local orthonormal frame field $\{E_1,\dots,E_m\}$. We then have $A(E_i)=\bar{k}_i E_i$, $i=1,\dots,m$, where

$$\bar{k}_1(q)=\cdots=\bar{k}_{m_1}(q)=k_1(q)\,,\qquad \bar{k}_{m_1+1}(q)=\cdots=\bar{k}_m(q)=k_2(q),$$

and $k_1(q)\neq k_2(q)$, for any $q\in U$. From (1.5) we can assume that

$$k_1=-\frac{m}{2}f \tag{1.21}$$

and $E_1=\operatorname{grad} f/|\operatorname{grad} f|$ on U.

Since $\langle E_\alpha,E_1\rangle=0$, we have on U

$$E_\alpha(f)=0\,,\quad \forall\,\alpha=2,\dots,m. \tag{1.22}$$

We shall use the connection equations with respect to the frame field $\{E_1,\dots,E_m\}$,

$$\nabla_{E_i}E_j=\omega_j^k(E_i)E_k. \tag{1.23}$$

Let us first prove that the multiplicity of k_1 is $m_1=1$. Suppose that $m_1\geq 2$. Then there exists $\alpha\in\{2,\dots,m_1\}$, such that $\bar{k}_\alpha=k_1$ on U. Since $\nabla^\perp\eta=0$, the Codazzi equation for A writes as

$$(\nabla_{E_i}A)(E_j)=(\nabla_{E_j}A)(E_i),\quad \forall\,i,j=1,\dots,m. \tag{1.24}$$

By using (1.23), the Codazzi equation becomes

$$E_i(\bar{k}_j)E_j+\sum_{\ell=1}^m(\bar{k}_j-\bar{k}_\ell)\omega_j^\ell(E_i)E_\ell=E_j(\bar{k}_i)E_i+\sum_{\ell=1}^m(\bar{k}_i-\bar{k}_\ell)\omega_i^\ell(E_j)E_\ell. \tag{1.25}$$

Putting $i=1$ and $j=\alpha$ in (1.25) and taking the scalar product with E_α we obtain $E_1(k_1)=0$, which, together with (1.21) and (1.22), gives $f=$ constant, and this is a contradiction.

Thus $\bar{k}_1=k_1$ and $\bar{k}_\alpha=k_2$, for all $\alpha=2,\dots,m$, and since $\operatorname{trace}A=mf$, we get

$$k_2=\frac{3}{2}\frac{m}{m-1}f. \tag{1.26}$$

Putting $i=1$ and $j=\alpha$ in (1.25) and taking the scalar product with E_α, E_β, $\beta\neq\alpha$, and E_1, respectively, one gets

$$\omega_1^\alpha(E_\alpha)=-\frac{3}{m+2}\frac{E_1(f)}{f}, \tag{1.27a}$$

$$\omega_1^\alpha(E_\beta)=0, \tag{1.27b}$$

$$\omega_1^\alpha(E_1)=0, \tag{1.27c}$$

for all $\alpha,\beta=2,\dots,m$, $\alpha\neq\beta$.

We now express the Gauss equation for U in $\mathbb{S}^n$,

$$\begin{aligned}\langle R^{\mathbb{S}^n}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle \\ &\quad+\langle B(X,Z),B(Y,W)\rangle-\langle B(X,W),B(Y,Z)\rangle,\end{aligned} \tag{1.28}$$

with $X = W = E_1$ and $Y = Z = E_\alpha$. One obtains

$$B(E_1, E_\alpha) = 0, \quad B(E_1, E_1) = k_1\eta, \quad \langle B(E_\alpha, E_\alpha), B(E_1, E_1)\rangle = k_1k_2.$$

From (1.23), (1.27b), (1.27c), and using $\omega_j^k = -\omega_k^j$, the curvature term is

$$\langle R(E_1, E_\alpha)E_\alpha, E_1\rangle = -E_1(\omega_1^\alpha(E_\alpha)) - (\omega_1^\alpha(E_\alpha))^2.$$

Finally, (1.28) and (1.27a) imply

$$fE_1(E_1(f)) = \frac{m+2}{3}f^2 - \frac{m^2(m+2)}{4(m-1)}f^4 + \frac{m+5}{m+2}(E_1(f))^2. \tag{1.29}$$

From (1.21) and (1.26), we have

$$|A|^2 = k_1^2 + (m-1)k_2^2 = \frac{m^2(m+8)}{4(m-1)}f^2. \tag{1.30}$$

Moreover, using (1.22), (1.23) and (1.27a), the Laplacian of f becomes

$$\begin{aligned} \Delta f &= -E_1(E_1(f)) - \sum_{\alpha=2}^m E_\alpha(E_\alpha(f)) + (\nabla_{E_1}E_1)f + \sum_{\alpha=2}^m (\nabla_{E_\alpha}E_\alpha)f \\ &= -E_1(E_1(f)) + \sum_{\alpha=2}^m \omega_\alpha^1(E_\alpha)E_1(f) \\ &= -E_1(E_1(f)) + \frac{3(m-1)}{m+2}\frac{(E_1(f))^2}{f}. \end{aligned} \tag{1.31}$$

From (1.5)(i), by substituting (1.30) and (1.31), we get

$$fE_1(E_1(f)) = -mf^2 + \frac{m^2(m+8)}{4(m-1)}f^4 + \frac{3(m-1)}{m+2}(E_1(f))^2. \tag{1.32}$$

Consider now $\gamma = \gamma(u)$ to be an arbitrary integral curve of E_1 in U. Along γ we have $f = f(u)$ and we set $w = (E_1(f))^2 = (f')^2$. Then $dw/df = 2f''$, and (1.29) and (1.32) become

$$\begin{cases} \frac{1}{2}f\frac{dw}{df} = \frac{m+2}{3}f^2 - \frac{m^2(m+2)}{4(m-1)}f^4 + \frac{m+5}{m+2}w, \\ \frac{1}{2}f\frac{dw}{df} = -mf^2 + \frac{m^2(m+8)}{4(m-1)}f^4 + \frac{3(m-1)}{m+2}w. \end{cases} \tag{1.33}$$

By subtracting the two equations we find two cases.

If $m = 4$, then

$$-\frac{2(2m+1)}{3f^2}f^2 + \frac{m^2(m+5)}{2(m-1)}f^4 = 0,$$

thus f is constant.

If $m \neq 4$, then

$$w = \frac{(m+2)(2m+1)}{3(m-4)} f^2 - \frac{m^2(m+2)(m+5)}{4(m-4)(m-1)} f^4.$$

Differentiating with respect to f and replacing this in the second equation of (1.33), we get

$$\frac{(m-1)(m+5)}{3} f^2 + \frac{3m^2(2m+1)}{4(m-1)} f^4 = 0.$$

Therefore f is constant along γ, thus $\operatorname{grad} f = 0$ along γ and we have a contradiction. □

Theorem 1.21 ([21]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a hypersurface. Assume that φ is proper-biharmonic with at most two distinct principal curvatures everywhere. Then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$. Moreover, if M is complete, then either $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ and φ is an embedding, or $\varphi(M) = \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$ and φ is an embedding when $m_1 \geq 2$ and $m_2 \geq 2$.*

Proof. By Theorem 1.20 and Theorem 1.7, the mean curvature of M in $\mathbb{S}^{m+1}$ is constant, $|H| \in (0,1]$. Thus we have a globally defined unit section in the normal bundle $\eta = H/|H|$ and a globally defined mean curvature function $f = |H|$. By using Corollary 1.2, we also obtain $|A|^2 = m$.

We now have two situations.

(i) If there exists an umbilical point $p_0 \in M$ and we denote by $k(p_0)$ the principal curvature with respect to η at p_0, then

$$mk(p_0) = m|H(p_0)|,$$

and $|A|^2 = m$ implies

$$mk^2(p_0) = m.$$

These two relations lead to $|H(p_0)| = 1$, but $|H|$ is constant, thus $|H| = 1$ on M. From Theorem 1.7 we conclude that $\varphi(M)$ is an open part of $\mathbb{S}^m(1/\sqrt{2})$.

(ii) If M has only non-umbilical points, we have the globally defined continuous principal curvature functions k_1 and k_2 with multiplicity functions m_1 and m_2 with respect to $\eta = H/|H|$, and $k_1(p) \neq k_2(p)$, for all $p \in M$. As discussed at the beginning of the proof of Theorem 1.20, m_1 and m_2 are constant on M. Since k_1 and k_2 are the solutions of

$$\begin{cases} m_1 k_1 + m_2 k_2 = mf, \\ \\ m_1 k_1^2 + m_2 k_2^2 = m, \end{cases} \tag{1.34}$$

where m_1, m_2 and f are constant, we conclude that M has two distinct constant principal curvatures. Theorem 1 in [115] implies that $\varphi(M)$ is an open part of the product of two spheres $\mathbb{S}^{m_1}(a) \times \mathbb{S}^{m_2}(b)$, such that $a^2 + b^2 = 1$, $m_1 + m_2 = m$. Since M is biharmonic in $\mathbb{S}^n$, from Theorem 1.6, we get that $a = b = 1/\sqrt{2}$ and $m_1 \neq m_2$.

The last statement of the theorem follows by a standard argument presented by K. Nomizu and B. Smyth in [105]. □

Corollary 1.22 ([30]). *Let $\varphi : M^2 \to \mathbb{S}^3$ be a proper-biharmonic surface. Then $\varphi(M)$ is an open part of $\mathbb{S}^2(1/\sqrt{2}) \subset \mathbb{S}^3$.*

Theorem 1.23 ([21]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$, $m \geq 3$, be a proper-biharmonic hypersurface. The following statements are equivalent:*

(i) φ is quasi-umbilical,

(ii) φ is conformally flat,

(iii) $\varphi(M)$ is an open part of $\mathbb{S}^m(1/\sqrt{2})$ or of $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.

Proof. By Theorem 1.21 we get that (i) is equivalent to (iii). Also, note that (iii) obviously implies (ii).

In order to prove that (ii) implies (i), recall that, for $m \geq 4$, by a well-known result (see, for example, [45]), any conformally flat hypersurface of a space form is quasi-umbilical and we conclude.

For $m = 3$, as the hypersurface is conformally flat, it follows that the $(0,2)$-tensor field $L = -\operatorname{Ricci} + \frac{s}{4}\langle\ ,\ \rangle$, where s is the scalar curvature of M, is a Codazzi tensor field, i.e.

$$(\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z), \quad \forall X,Y,Z \in C(TM). \tag{1.35}$$

Using the notations from the proof of Theorem 1.20, the Gauss equation implies

$$\operatorname{Ricci}(X,Y) = 2\langle X,Y\rangle + 3f\langle A(X),Y\rangle - \langle A(X),A(Y)\rangle$$

and

$$s = 6 + 9f^2 - |A|^2. \tag{1.36}$$

We use the same techniques as in the proof of Theorem 1.20. Suppose the existence of an open subset U of M with 3 distinct principal curvatures.

If f is constant on U, using the above expressions, we conclude that U is flat and that the product of any of its two principal curvatures is -1, thus we get to a contradiction.

Assume that f is not constant on U. We can suppose that $\operatorname{grad}_p f \neq 0$, $\forall p \in U$. Consider $E_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$. As M is proper biharmonic, E_1 gives a principal direction with principal curvature $k_1 = -\dfrac{3}{2}f$. From $k_1 + k_2 + k_3 = 3f$, we can write $k_2 = \dfrac{9}{4}f + \varepsilon$ and $k_3 = \dfrac{9}{4}f - \varepsilon$, $\varepsilon \in C^\infty(U)$. Using the Codazzi and Gauss equations and equations (1.35) and (1.36) we show that $f = a\varepsilon^5$, $a \in \mathbb{R}$, and combining all these relations we obtain that ε is a solution of a polynomial equation with constant coefficients. Thus ε and f are constant.

Hence M has at most two distinct principal curvatures and this completes the proof. □

It is well known that, if $m \geq 4$, a hypersurface $\varphi : M^m \to \mathbb{S}^{m+1}$ is quasi-umbilical if and only if it is conformally flat. From Theorem 1.23 we see that under the biharmonicity hypothesis the equivalence remains true when $m = 3$.

We recall that an orientable hypersurface $\varphi : M^m \to \mathbb{S}^{m+1}$ is said to be *isoparametric* if it has constant principal curvatures or, equivalently, the number ℓ of distinct principal curvatures $k_1 > k_2 > \cdots > k_\ell$ is constant on M and the k_i's are constant. The distinct principal curvatures have constant multiplicities $m_1, \ldots, m_\ell$, $m = m_1 + m_2 + \ldots + m_\ell$.

The isoparametric hypersurfaces with $\ell \leq 3$ were studied by E. Cartan (see [36]) and P.J. Ryan (see [116]). We present here a general result.

Theorem 1.24 ([36, 116])**.** *Let $\varphi : M \to \mathbb{S}^{m+1}$ be an m-dimensional isoparametric hypersurface in $\mathbb{S}^{m+1}$. Let $k_1 > k_2 > \ldots > k_\ell$ be the distinct principal curvatures with multiplicities $m_1, \ldots, m_\ell$, $m = m_1 + m_2 + \ldots + m_\ell$. Then*

(i) *ℓ is either $1, 2, 3, 4$ or 6.*

(ii) *If $\ell = 1$, then M is totally umbilical.*

(iii) *If $\ell = 2$, then $\varphi(M)$ is an open part of $\mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$, $r_1^2 + r_2^2 = 1$.*

(iv) *If $\ell = 3$, then $m_1 = m_2 = m_3 = 2^q$, $q = 0, \ldots, 3$.*

(v) *If $\ell = 4$, $m_1 = m_3$ and $m_2 = m_4$. Moreover, $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $m_1 + m_2 + 1$ is a multiple of $2^{\zeta(m^*-1)}$. Here $\zeta(n)$ is the number of integers a with $1 < a < n$, $a \equiv 0, 1, 2, 4 \mod 8$ and $m^* = \min\{m_1, m_2\}$.*

(vi) *If $\ell = 6$, $m_1 = m_2 = \ldots = m_6 = 1$ or 2.*

(vii) *There exists an angle θ, $0 < \theta < \frac{\pi}{\ell}$, such that*

$$k_\alpha = \cot\big(\theta + (\alpha - 1)\pi/\ell\big), \quad \alpha = 1, \ldots, \ell.$$

The next result on hypersurfaces with 3 distinct principal curvatures was proved.

Theorem 1.25 ([37])**.** *A compact hypersurface M^m of constant scalar curvature s and constant mean curvature $|H|$ in $\mathbb{S}^{m+1}$ is isoparametric provided it has 3 distinct principal curvatures everywhere.*

In order to analyze the case of $\mathbb{S}^4$, we shall need the following.

Theorem 1.26 ([51])**.** *Any complete hypersurface with constant scalar and mean curvature in $\mathbb{S}^4$ is isoparametric.*

For what concerns biharmonic hypersurfaces with 3 distinct principal curvatures in spheres, the following non-existence result was proved.

Theorem 1.27 ([19])**.** *There exist no compact CMC proper-biharmonic hypersurfaces $\varphi : M^m \to \mathbb{S}^{m+1}$ with three distinct principal curvatures everywhere.*

Proof. From the hypothesis it follows that M has constant scalar curvature. Since M has three distinct principal curvatures, we can apply Theorem 1.25 and we get that M is isoparametric with $\ell = 3$ in $\mathbb{S}^{m+1}$.

We now use (vii) in Theorem 1.24 in order to express the principal curvatures of M. There exists $\theta \in (0, \pi/3)$ such that

$$k_1 = \cot\theta, \qquad k_2 = \cot\big(\theta + \frac{\pi}{3}\big) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \qquad k_3 = \cot\big(\theta + \frac{2\pi}{3}\big) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}.$$

Thus,

$$|A|^2 = 2^q(k_1^2 + k_2^2 + k_3^2) = 2^q\,\frac{9k_1^6 + 45k_1^2 + 6}{(1 - 3k_1^2)^2}. \tag{1.37}$$

Moreover, from (iv) in Theorem 1.24 we obtain $m = 3 \cdot 2^q$, $q = 0, \ldots, 3$ and since M is biharmonic of constant mean curvature, from (1.5), we get $|A|^2 = m = 3 \cdot 2^q$.

The last equation together with (1.37) implies that k_1^2 is a solution of $P(t) = 3t^3 - 9t^2 + 21t + 1 = 0$, which is an equation with no positive roots. Indeed, $P(0) = 1 > 0$ and $P'(t) = 9t^2 - 18t + 21 > 0$, for all $t \in \mathbb{R}$, hence P is an increasing function on $\mathbb{R}$. $\square$

Then, in [77, 78], T. Ichiyama, J.I. Inoguchi and H. Urakawa classified all proper-biharmonic isoparametric hypersurfaces in spheres.

Theorem 1.28 ([77, 78])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be an orientable isoparametric hypersurface. If φ is proper-biharmonic, then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

The biharmonic hypersurfaces in $\mathbb{S}^4$ were studied in [19].

Theorem 1.29 ([19])**.** *Let $\varphi : M^3 \to \mathbb{S}^4$ be a proper-biharmonic hypersurface. Then φ is CMC.*

Proof. Suppose that $|H|$ is not constant on M. Then there exists an open subset U of M such that $\operatorname{grad}_p |H|^2 \neq 0$, for all $p \in U$. By eventually restraining U we can suppose that $|H| > 0$ on U, and thus $\operatorname{grad}_p |H| \neq 0$, for all $p \in U$. If U has at most two distinct principal curvatures, then , by Theorem 4.1 in [21], we conclude that its mean curvature is constant and we have a contradiction. Then there exists a point in U with three distinct principal curvatures. This implies the existence of an open neighborhood of points with three distinct principal curvatures and we can suppose, by restraining U, that all its points have three distinct principal curvatures. On U we can consider the unit section in the normal bundle $\eta = H/|H|$ and denote by $f = |H|$ the mean curvature function of U in $\mathbb{S}^{m+1}(c)$ and by k_i, $i = 1, 2, 3$, its principal curvatures w.r.t. η.

Conclusively, the hypothesis for M to be proper-biharmonic with at most three distinct principal curvatures in $\mathbb{S}^{m+1}(c)$ and non-constant mean curvature, implies the existence of an open connected subset U of M, with

$$\begin{cases} \operatorname{grad}_p f \neq 0, \\[2ex] f(p) > 0, \\[2ex] k_1(p) \neq k_2(p) \neq k_3(p) \neq k_1(p), \qquad \forall p \in U. \end{cases} \tag{1.38}$$

We shall contradict the condition $\operatorname{grad}_p f \neq 0$, for all $p \in U$.

Since M is proper-biharmonic in $\mathbb{S}^4(c)$, from (1.5) we have

$$\begin{cases} \Delta f = (3c - |A|^2) f, \\ \\ A(\operatorname{grad} f) = -\dfrac{3}{2} f \operatorname{grad} f. \end{cases} \tag{1.39}$$

We can consider $k_1 = -\frac{3}{2} f$ and $X_1 = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$ on U. Then X_1 is a principal direction corresponding to the principal curvature k_1. Recall that $3f = k_1 + k_2 + k_3$, thus

$$k_2 + k_3 = \frac{9}{2} f. \tag{1.40}$$

We shall use the moving frames method and denote by X_1, X_2, X_3 the orthonormal frame field of principal directions and by $\{\omega^a\}_{a=1}^3$ the dual frame field of $\{X_a\}_{a=1}^3$ on U.

Obviously,

$$X_i(f) = \langle X_i, \operatorname{grad} f\rangle = |\operatorname{grad} f| \langle X_i, X_1\rangle = 0, \qquad i = 2, 3, \tag{1.41}$$

thus

$$\operatorname{grad} f = X_1(f) X_1. \tag{1.42}$$

We write

$$\nabla X_a = \omega_a^b X_b, \qquad \omega_a^b \in C(T^*U).$$

From the Codazzi equations for M, for distinct $a, b, d = 1, 2, 3$, we get

$$X_a(k_b) = (k_a - k_b)\omega_a^b(X_b) \tag{1.43}$$

and

$$(k_b - k_d)\omega_b^d(X_a) = (k_a - k_d)\omega_a^d(X_b). \tag{1.44}$$

Consider now in (1.43), $a = 1$ and $b = i$ and, respectively, $a = i$ and $b = j$ with $i \neq j$. We obtain

$$\omega_i^1(X_i) \quad = \quad \frac{X_1(k_i)}{k_i - k_1}$$

and

$$\omega_j^i(X_j) \quad = \quad \frac{X_i(k_j)}{k_j - k_i}.$$

For $a = i$ and $b = 1$, as $X_i(k_1) = 0$, (1.43) leads to $\omega_i^1(X_1) = 0$ and we can write

$$\omega_a^1(X_1) = 0, \quad a = 1, 2, 3.$$

Notice that, since $X_i(f) = 0$, then $\langle [X_i, X_j], X_1\rangle = 0$, thus $\omega_1^j(X_i) = \omega_1^i(X_j)$. Now, from (1.44), for $a = 1$, $b = i$ and $d = j$, with $i \neq j$, we get

$$\omega_2^1(X_3) = \omega_3^2(X_1) = \omega_1^3(X_2) = 0.$$

The structure 1-forms are thus determined by the following set of relations

$$\begin{cases} \omega_2^1(X_1)=0, & \omega_2^1(X_2)=\frac{X_1(k_2)}{k_2+\frac{3}{2}f}=\alpha_2, & \omega_2^1(X_3)=0, \\ \omega_3^1(X_1)=0, & \omega_3^1(X_2)=0, & \omega_3^1(X_3)=\frac{X_1(k_3)}{k_3+\frac{3}{2}f}=\alpha_3, \\ \omega_3^2(X_1)=0, & \omega_3^2(X_2)=\frac{X_3(k_2)}{k_3-k_2}=\beta_2, & \omega_3^2(X_3)=\frac{X_2(k_3)}{k_3-k_2}=\beta_3, \end{cases} \tag{1.45}$$

In order to express the first condition in (1.39), by using (1.40), we compute

$$\begin{aligned} |A|^2 &= k_1^2+k_2^2+k_3^2 \\ &= k_1^2+(k_2+k_3)^2-2k_2k_3 \\ &= \frac{45}{2}f^2-2K, \end{aligned} \tag{1.46}$$

where K denotes the product k_2k_3. From (1.42) we deduce that

$$\begin{aligned} \Delta f &= -\operatorname{div}(\operatorname{grad} f) = -\operatorname{div}(X_1(f)X_1) = -X_1(X_1(f)) - X_1(f)\operatorname{div} X_1 \\ &= -X_1(X_1(f)) + X_1(f)(\omega_2^1(X_2)+\omega_3^1(X_3)) \\ &= -X_1(X_1(f)) + X_1(f)(\alpha_2+\alpha_3). \end{aligned} \tag{1.47}$$

Now, by using (1.46) and (1.47), the equation $\Delta f = (3c-|A|^2)f$ becomes

$$X_1(X_1(f)) - X_1(f)(\alpha_2+\alpha_3) + (2K+3c-\frac{45}{2}f^2)f = 0. \tag{1.48}$$

We also compute

$$\begin{aligned} [X_1,X_i] &= \nabla_{X_1}X_i - \nabla_{X_i}X_1 = \langle \nabla_{X_1}X_i, X_1\rangle X_1 - \langle \nabla_{X_i}X_1, X_i\rangle X_i \\ &= \omega_i^1(X_i)X_i = \alpha_i X_i. \end{aligned} \tag{1.49}$$

We shall now use the Gauss equation

$$\begin{aligned} \langle R^{\mathbb{S}^4}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle \\ &\quad +\langle B(X,Z),B(Y,W)\rangle - \langle B(X,W),B(Y,Z)\rangle. \end{aligned} \tag{1.50}$$

From (1.50) we have:

- for $X=W=X_1$ and $Y=Z=X_i$

$$\begin{cases} X_1(\alpha_2) = \alpha_2^2 + c - \frac{3}{2}fk_2, \\ X_1(\alpha_3) = \alpha_3^2 + c - \frac{3}{2}fk_3; \end{cases} \tag{1.51}$$

- for $X=W=X_2$ and $Y=Z=X_3$

$$K + c = X_2(\beta_3) - X_3(\beta_2) - \alpha_2\alpha_3 - \beta_2^2 - \beta_3^2; \tag{1.52}$$

- for $Y = W = X_3$, $X = X_2$ and $Z = X_1$ and, respectively, for $X = W = X_2$, $Y = X_3$ and $Z = X_1$

$$\begin{cases} X_2(\alpha_3) = \beta_3(\alpha_3 - \alpha_2), \\ \\ X_3(\alpha_2) = \beta_2(\alpha_3 - \alpha_2); \end{cases} \tag{1.53}$$

- for $X = W = X_2$, $Y = X_1$ and $Z = X_3$, and, respectively, for $X = W = X_3$, $Y = X_2$ and $Z = X_1$

$$\begin{cases} X_1(\beta_2) = \alpha_2\beta_2, \\ \\ X_1(\beta_3) = \alpha_3\beta_3. \end{cases} \tag{1.54}$$

Notice now that, from (1.41) and (1.49),

$$X_i(X_1(f)) = -[X_1, X_i]f + X_1(X_i(f)) = -\alpha_i X_i(f) + X_1(X_i(f)) = 0 \tag{1.55}$$

and

$$X_i(X_1(X_1(f))) = 0. \tag{1.56}$$

Also, since $K = \dfrac{(k_2 + k_3)^2 - (k_3 - k_2)^2}{4}$ we obtain

$$\begin{cases} X_2(K) = -(k_3 - k_2)^2\beta_3, \\ \\ X_3(K) = (k_3 - k_2)^2\beta_2. \end{cases} \tag{1.57}$$

We differentiate (1.48) along X_2 and X_3 and use (1.53), (1.55), (1.56) and (1.57). We get

$$\begin{cases} X_2(\alpha_2) = -\beta_3(\alpha_3 - \alpha_2) - \dfrac{2f}{X_1(f)}(k_3 - k_2)^2\beta_3, \\ \\ X_3(\alpha_3) = -\beta_2(\alpha_3 - \alpha_2) + \dfrac{2f}{X_1(f)}(k_3 - k_2)^2\beta_2. \end{cases} \tag{1.58}$$

We intend to prove that $X_i(k_j) = 0$, $i, j = 2, 3$. In order to do this we apply $[X_1, X_2] = \alpha_2 X_2$ to the quantity α_2. On one hand, from (1.58), we get

$$[X_1, X_2]\alpha_2 = \alpha_2 X_2(\alpha_2) = \beta_3\Big\{ - \alpha_2\alpha_3 + \alpha_2^2 - \frac{2f}{X_1(f)}(k_3 - k_2)^2\alpha_2\Big\}. \tag{1.59}$$

On the other hand, by using (1.51) and (1.58), we obtain

$$\begin{aligned} [X_1, X_2]\alpha_2 &= X_1(X_2(\alpha_2)) - X_2(X_1(\alpha_2)) \\ &= \beta_3\Big\{ - 2\alpha_3^2 - \alpha_2^2 + 3\alpha_2\alpha_3 + \frac{2f}{X_1(f)}\big[- 2(k_3 - k_2)X_1(k_3 - k_2) \\ &\quad +(k_3 - k_2)^2(2\alpha_2 - \alpha_3)\big] - 2X_1\Big(\frac{f}{X_1(f)}\Big)(k_3 - k_2)^2\Big\}. \end{aligned} \tag{1.60}$$

By putting together (1.59) and (1.60) we either have $\beta_3 = 0$ or

$$X_1\Big(\frac{f}{X_1(f)}\Big) = -\frac{(\alpha_3 - \alpha_2)^2}{(k_3 - k_2)^2} + \frac{f}{X_1(f)}\Big(3\alpha_2 - \alpha_3 - 2\frac{X_1(k_3 - k_2)}{k_3 - k_2}\Big). \tag{1.61}$$

Moreover,

$$\begin{aligned} X_2\Big(\frac{X_1(k_3-k_2)}{k_3-k_2}\Big) &= -\frac{1}{k_3-k_2}[X_1,X_2](k_3-k_2)+X_1\Big(\frac{X_2(k_3-k_2)}{k_3-k_2}\Big) \\ &= 2(\alpha_3-\alpha_2)\beta_3. \end{aligned}$$

Suppose that $\beta_3\neq 0$, differentiate (1.61) along X_2 and use (1.53) and (1.58). We get

$$2(\alpha_3-\alpha_2)=-\frac{f}{X_1(f)}(k_3-k_2)^2. \tag{1.62}$$

We differentiate now (1.62) along X_2 and obtain

$$\alpha_3-\alpha_2=-2\frac{f}{X_1(f)}(k_3-k_2)^2, \tag{1.63}$$

and since $k_2\neq k_3$ the equations (1.62) and (1.63) lead to a contradiction.

Analogously, by using the symmetry of the equations in X_2 and X_3, we immediately prove that $\beta_2=0$.

We rewrite equations (1.51) in the form

$$\begin{cases} X_1(X_1(k_2))=\frac{21}{2}\alpha_2X_1(f)+2(K+c)(k_3+\frac{3}{2}f)+(c-\frac{3}{2}fk_2)(k_2+\frac{3}{2}f), \\ X_1(X_1(k_3))=\frac{21}{2}\alpha_3X_1(f)+2(K+c)(k_2+\frac{3}{2}f)+(c-\frac{3}{2}fk_3)(k_3+\frac{3}{2}f), \end{cases} \tag{1.64}$$

and by summing up we obtain

$$X_1(X_1(f))=\frac{7}{3}X_1(f)(\alpha_2+\alpha_3)+f(4K+5c-9f^2). \tag{1.65}$$

Now, by using (1.48) and (1.65) we obtain

$$X_1(f)(\alpha_2+\alpha_3)=f\big(-\frac{9}{2}K-6c+\frac{189}{8}f^2\big). \tag{1.66}$$

We replace (1.66) in (1.65) and get

$$X_1(X_1(f))=f\big(-\frac{13}{2}K-9c+\frac{369}{8}f^2\big). \tag{1.67}$$

In order to get another relation on f and K we first use (1.52), (1.51), (1.40), (1.45) and determine

$$\begin{aligned} X_1(K) &= -X_1(\alpha_2\alpha_3) \qquad (1.68)\\ &= -(\alpha_2\alpha_3+c)(\alpha_2+\alpha_3)+\frac{3}{2}f(\alpha_2k_3+\alpha_3k_2) \\ &= (K+9f^2)(\alpha_2+\alpha_3)-\frac{27}{4}fX_1(f). \end{aligned}$$

By differentiating (1.66) along X_1, and by using (1.68), (1.67), (1.51), (1.66) we get

$$X_1(f)\Big(\frac{13}{2}K+10c-108f^2\Big)=f(\alpha_2+\alpha_3)\Big(\frac{13}{2}K+15c-\frac{441}{4}f^2\Big). \tag{1.69}$$

We multiply (1.69) first by $X_1(f)$ and secondly by $\alpha_2+\alpha_3$ and, by using (1.66), we get

$$\begin{cases} (X_1(f))^2(\frac{13}{2}K+10c-108f^2) = f^2(-\frac{9}{2}K-6c+\frac{189}{8}f^2)(\frac{13}{2}K+15c-\frac{441}{4}f^2), \\ (\frac{13}{2}K+10c-108f^2)(-\frac{9}{2}K-6c+\frac{189}{8}f^2) = (\alpha_2+\alpha_3)^2(\frac{13}{2}K+15c-\frac{441}{4}f^2). \end{cases} \tag{1.70}$$

Differentiating (1.66) along X_1, and using (1.68), (1.67), (1.51), (1.66), (1.70), we obtain

$$\begin{aligned} 27f^2(4044800c^3 - 49579440c^2f^2 + 187840944cf^4 - 254205945f^6) \\ -6(51200c^3 - 19600320c^2f^2 + 119328660cf^4 - 80969301f^6)K \\ -208(2240c^2 - 108396cf^2 - 285363f^4)K^2 \\ +2704(16c - 2277f^2)K^3 \\ +140608K^4 = 0. \end{aligned} \tag{1.71}$$

Consider now $\gamma = \gamma(t)$, $t \in I$, to be an integral curve of X_1 passing through $p = \gamma(t_0)$. Since $X_2(f) = X_3(f) = 0$ and $X_2(K) = X_3(K) = 0$ and $X_1(f) \neq 0$, we can write $t = t(f)$ in a neighborhood of $f_0 = f(t_0)$ and thus consider $K = K(f)$.

Notice that if $\frac{13}{2}K + 15c - \frac{441}{4}f^2 = 0$ or $10c + \frac{13}{2}K - 108f^2 = 0$, then from (1.71) the function f results to be the solution of a polynomial equation of eighth degree with constant coefficients and we would get to a contradiction. Thus, from (1.70) we have that

$$\begin{cases} (\frac{df}{dt})^2 = \dfrac{f^2(-\frac{9}{2}K-6c+\frac{189}{8}f^2)(\frac{13}{2}K+15c-\frac{441}{4}f^2)}{\frac{13}{2}K+10c-108f^2}, \\ (\alpha_2+\alpha_3)^2 = \frac{(\frac{13}{2}K+10c-108f^2)(-\frac{9}{2}K-6c+\frac{189}{8}f^2)}{\frac{13}{2}K+15c-\frac{441}{4}f^2}. \end{cases} \tag{1.72}$$

We can now compute $\frac{dK}{df}$ by using (1.72), (1.68) and (1.66),

$$\begin{aligned} \frac{dK}{df} &= \frac{dK}{dt}\frac{dt}{df} \\ &= \frac{(K+9f^2)\frac{df}{dt}(\alpha_2+\alpha_3)}{(\frac{df}{dt})^2} - \frac{27}{4}f \\ &= \frac{(K+9f^2)(\frac{13}{2}K+10c-108f^2)}{f(\frac{13}{2}K+15c-\frac{441}{4}f^2)} - \frac{27}{4}f. \end{aligned} \tag{1.73}$$

The next step consists in differentiating (1.71) with respect to f. By substituting $\dfrac{dK}{df}$ from (1.73) we get another polynomial equation in f and K, of fifth degree in K. We eliminate K^5 between this new polynomial equation and (1.71). The result constitutes a polynomial equation in f and K, of fourth degree in K. In a similar way, by using (1.71) and its consequences we are able to gradually eliminate K^4, K^3, K^2 and K and we are led to a polynomial equation with constant coefficients in f. Thus f results to be constant and we conclude. □

Theorem 1.30. *Let $\varphi : M^3 \to \mathbb{S}^4$ be a complete proper-biharmonic hypersurface. Then $\varphi(M) = \mathbb{S}^3(1/\sqrt{2})$ or $\varphi(M) = \mathbb{S}^2(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Proof. Suppose that M^3 is a compact proper-biharmonic hypersurface of $\mathbb{S}^4$. From Theorem 1.29 follows that M has constant mean curvature and, since it satisfies the hypotheses of Theorem 1.17, we conclude that it also has constant scalar curvature. We can thus apply Theorem 1.26 and it results that M is isoparametric in $\mathbb{S}^4$. From Theorem 1.27 we get that M cannot be isoparametric with $\ell = 3$, and by using Theorem 1.21 we conclude the proof. □

An orientable hypersurface $\varphi : M^m \to \mathbb{S}^{m+1}$ is said to be a *proper Dupin hypersurface* if the number ℓ of distinct principal curvatures is constant on M and each principal curvature function is constant along its corresponding principal directions.

Theorem 1.31 ([16])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be an orientable proper Dupin hypersurface. If φ is proper-biharmonic, then φ is CMC.*

Proof. As M is orientable, we fix $\eta \in C(NM)$ and denote $A = A_\eta$ and $f = (\operatorname{trace} A)/m$.

Suppose that f is not constant. Then there exists an open subset $U \subset M$ such that $\operatorname{grad} f \neq 0$ at every point of U. Since φ is proper-biharmonic, from (1.5) we get that $-mf/2$ is a principal curvature with principal direction $\operatorname{grad} f$. Since the hypersurface is proper Dupin, by definition, $\operatorname{grad} f(f) = 0$, i.e. $\operatorname{grad} f = 0$ on U, and we come to a contradiction. □

Corollary 1.32 ([16])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be an orientable proper Dupin hypersurface with $\ell \leq 3$. If φ is proper-biharmonic, then $\varphi(M)$ is either an open part of $\mathbb{S}^m(1/\sqrt{2})$, or an open part of $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Proof. Taking into account Theorem 1.21, we only have to prove that there exist no proper-biharmonic proper Dupin hypersurfaces with $\ell = 3$. Indeed, by Theorem 1.31, we conclude that φ is CMC. By a result in [17], φ is of type 1 or of type 2, in the sense of B.-Y. Chen. If φ is of type 1, we must have $\ell = 1$ and we get a contradiction. If φ is of type 2, since φ is proper Dupin with $\ell = 3$, from Theorem 9.11 in [42], we get that φ is isoparametric. But, from Theorem 1.28, proper-biharmonic isoparametric hypersurfaces must have $\ell \leq 2$.

□

1.4.2 Case 2

The simplest result is the following.

Proposition 1.33 ([16])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact hypersurface. Assume that φ is proper-biharmonic with nowhere zero mean curvature vector field and $|A|^2 \leq m$, or $|A|^2 \geq m$. Then φ is CMC and $|A|^2 = m$.*

Proof. As H is nowhere zero, we can consider $\eta = H/|H|$ a global unit section in the normal bundle NM of M in $\mathbb{S}^{m+1}$. Then, on M,

$$\Delta f = (m - |A|^2)f,$$

where $f = (\operatorname{trace} A)/m = |H|$. Now, as $m - |A|^2$ does not change sign, from the maximum principle we get $f = \text{constant}$ and $|A|^2 = m$. □

In fact, Proposition 1.33 holds without the hypothesis "H nowhere zero". In order to prove this we shall consider the cases $|A|^2 \geq m$ and $|A|^2 \leq m$, separately.

Proposition 1.34 ([16])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact hypersurface. Assume that φ is proper-biharmonic and $|A|^2 \geq m$. Then φ is CMC and $|A|^2 = m$.*

Proof. Locally,

$$\Delta f = (m - |A|^2) f,$$

where $f = (\operatorname{trace} A)/m$, $f^2 = |H|^2$, and therefore

$$\frac{1}{2}\Delta f^2 = (m - |A|^2) f^2 - |\operatorname{grad} f|^2 \leq 0.$$

As f^2, $|A|^2$ and $|\operatorname{grad} f|^2$ are well defined on the whole M, the formula holds on M. From the maximum principle we get that $|H|$ is constant and $|A|^2 = m$. □

The case $|A|^2 \leq m$ was solved by J.H. Chen in [48]. Here we include the proof for two reasons. First, the original one is in Chinese and second, the formalism used by J.H. Chen was local, while ours is globally invariant. Moreover, the proof we present is slightly shorter.

Theorem 1.35 ([48])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact hypersurface in $\mathbb{S}^{m+1}$. If φ is proper-biharmonic and $|A|^2 \leq m$, then φ is CMC and $|A|^2 = m$.*

Proof. We may assume that M is orientable, since, otherwise, we consider the double covering $\tilde{M}$ of M. This is compact, connected and orientable, and in the given hypotheses $\tilde{\varphi} : \tilde{M} \to \mathbb{S}^{m+1}$ is proper-biharmonic and $|\tilde{A}|^2 \leq m$. Moreover, $\tilde{\varphi}(\tilde{M}) = \varphi(M)$.

As M is orientable, we fix a unit global section $\eta \in C(NM)$ and denote $A = A_\eta$ and $f = (\operatorname{trace} A)/m$. In the following we shall prove that

$$\begin{aligned} &\frac{1}{2}\Delta\left(|\operatorname{grad} f|^2 + \frac{m^2}{8} f^4 + f^2\right) + \frac{1}{2}\operatorname{div}(|A|^2 \operatorname{grad} f^2) \leq \\ &\leq \frac{8(m-1)}{m(m+8)}(|A|^2 - m)|A|^2 f^2, \end{aligned} \tag{1.74}$$

on M, and this will lead to the conclusion.

From (1.5)(i) one easily gets

$$\frac{1}{2}\Delta f^2 = (m - |A|^2) f^2 - |\operatorname{grad} f|^2 \tag{1.75}$$

and

$$\frac{1}{4}\Delta f^4 = (m - |A|^2) f^4 - 3 f^2 |\operatorname{grad} f|^2. \tag{1.76}$$

From the Weitzenböck formula we have

$$\frac{1}{2}\Delta|\operatorname{grad} f|^2 = -\langle \operatorname{trace} \nabla^2 \operatorname{grad} f, \operatorname{grad} f\rangle - |\nabla \operatorname{grad} f|^2, \tag{1.77}$$

and, since

$$\operatorname{trace} \nabla^2 \operatorname{grad} f = -\operatorname{grad}(\Delta f) + \operatorname{Ricci}(\operatorname{grad} f),$$

we obtain

$$\frac{1}{2}\Delta|\operatorname{grad} f|^2 = \langle \operatorname{grad}\Delta f, \operatorname{grad} f\rangle - \operatorname{Ricci}(\operatorname{grad} f, \operatorname{grad} f) - |\nabla \operatorname{grad} f|^2. \tag{1.78}$$

Equations (1.5)(i) and (1.75) imply

$$\begin{aligned}\langle \operatorname{grad}\Delta f, \operatorname{grad} f\rangle &= (m-|A|^2)|\operatorname{grad} f|^2 - \frac{1}{2}\langle \operatorname{grad}|A|^2, \operatorname{grad} f^2\rangle \\ &= (m-|A|^2)|\operatorname{grad} f|^2 - \frac{1}{2}\left(\operatorname{div}(|A|^2 \operatorname{grad} f^2) + |A|^2\Delta f^2\right) \\ &= m|\operatorname{grad} f|^2 - \frac{1}{2}\operatorname{div}(|A|^2\operatorname{grad} f^2) - |A|^2(m-|A|^2)f^2. \end{aligned}\tag{1.79}$$

From the Gauss equation of M in $\mathbb{S}^{m+1}$ we obtain

$$\operatorname{Ricci}(X,Y) = (m-1)\langle X,Y\rangle + \langle A(X),Y\rangle \operatorname{trace} A - \langle A(X), A(Y)\rangle, \tag{1.80}$$

for all $X, Y \in C(TM)$, therefore, by using (1.5)(ii),

$$\operatorname{Ricci}(\operatorname{grad} f, \operatorname{grad} f) = \left(m - 1 - \frac{3m^2}{4}f^2\right)|\operatorname{grad} f|^2. \tag{1.81}$$

Now, by substituting (1.79) and (1.81) in (1.78) and using (1.75) and (1.76), one obtains

$$\begin{aligned}\frac{1}{2}\Delta|\operatorname{grad} f|^2 &= \left(1+\frac{3m^2}{4}f^2\right)|\operatorname{grad} f|^2 - \frac{1}{2}\operatorname{div}(|A|^2\operatorname{grad} f^2) \\ &\quad -|A|^2(m-|A|^2)f^2 - |\nabla \operatorname{grad} f|^2 \\ &= -\frac{1}{2}\Delta f^2 - \frac{m^2}{16}\Delta f^4 - (m-|A|^2)\left(|A|^2 - \frac{m^2}{4}f^2 - 1\right)f^2 \\ &\quad -\frac{1}{2}\operatorname{div}(|A|^2\operatorname{grad} f^2) - |\nabla \operatorname{grad} f|^2.\end{aligned}$$

Hence

$$\begin{aligned}&-\tfrac{1}{2}\Delta\left(|\operatorname{grad} f|^2 + \tfrac{m^2}{8}f^4 + f^2\right) - \tfrac{1}{2}\operatorname{div}(|A|^2\operatorname{grad} f^2) = \\ &= (m-|A|^2)\left(|A|^2 - \tfrac{m^2}{4}f^2 - 1\right)f^2 + |\nabla \operatorname{grad} f|^2.\end{aligned}\tag{1.82}$$

We shall now verify that

$$(m-|A|^2)\left(|A|^2 - \frac{m^2}{4}f^2 - 1\right) \geq (m-|A|^2)\left(\frac{9}{m+8}|A|^2 - 1\right), \tag{1.83}$$

at every point of M. Let us now fix a point $p \in M$. We have two cases.
Case 1. If $\operatorname{grad}_p f \neq 0$, then $e_1 = (\operatorname{grad}_p f)/|\operatorname{grad}_p f|$ is a principal direction for A with principal curvature $\lambda_1 = -mf(p)/2$. By considering $e_k \in T_pM$, $k = 2, \dots, m$, such that $\{e_i\}_{i=1}^m$ is an orthonormal basis in T_pM and $A(e_k) = \lambda_k e_k$, we get at p

$$\begin{aligned}|A|^2 &= \sum_{i=1}^m |A(e_i)|^2 = |A(e_1)|^2 + \sum_{k=2}^m |A(e_k)|^2 = \frac{m^2}{4}f^2 + \sum_{k=2}^m \lambda_k^2 \\ &\geq \frac{m^2}{4}f^2 + \frac{1}{m-1}\left(\sum_{k=2}^m \lambda_k\right)^2 = \frac{m^2(m+8)}{4(m-1)}f^2,\end{aligned}\tag{1.84}$$

thus inequality (1.83) holds at p.
Case 2. If $\operatorname{grad}_p f = 0$, then either there exists an open set $U \subset M$, $p \in U$, such that $\operatorname{grad} f_{/U} = 0$, or p is a limit point for the set $V = \{q \in M : \operatorname{grad}_q f \neq 0\}$.
In the first situation, we get that f is constant on U, and from a unique continuation result for biharmonic maps (see [108]), this constant must be different from zero. Equation (1.5)(i) implies $|A|^2 = m$ on U, and therefore inequality (1.83) holds at p.
In the second situation, by taking into account *Case 1* and passing to the limit, we conclude that inequality (1.83) holds at p.

In order to evaluate the term $|\nabla \operatorname{grad} f|^2$ of equation (1.82), let us consider a local orthonormal frame field $\{E_i\}_{i=1}^m$ on M. Then, also using (1.5)(i),

$$\begin{aligned} |\nabla \operatorname{grad} f|^2 &= \sum_{i,j=1}^m \langle \nabla_{E_i} \operatorname{grad} f, E_j\rangle^2 \geq \sum_{i=1}^m \langle \nabla_{E_i} \operatorname{grad} f, E_i\rangle^2 \\ &\geq \frac{1}{m}\left(\sum_{i=1}^m \langle \nabla_{E_i} \operatorname{grad} f, E_i\rangle\right)^2 = \frac{1}{m}(\Delta f)^2 \\ &= \frac{1}{m}(m - |A|^2)^2 f^2. \end{aligned} \tag{1.85}$$

In fact, (1.85) is a global formula.

Now, using (1.83) and (1.85) in (1.82), we obtain (1.74), and by integrating it, since $|A|^2 \leq m$, we get

$$(|A|^2 - m)|A|^2 f^2 = 0 \tag{1.86}$$

on M. Suppose that there exists $p \in M$ such that $|A(p)|^2 \neq m$. Then there exists an open set $U \subset M, p \in U$, such that $|A|^2_{/U} \neq m$. Equation (1.86) implies that $|A|^2 f^2_{/U} = 0$. Now, if there were a $q \in U$ such that $f(q) \neq 0$, then $A(q)$ would be zero and, therefore, $f(q) = 0$. Thus $f_{/U} = 0$ and, since M is proper-biharmonic, this is a contradiction. Thus $|A|^2 = m$ on M and $\Delta f = 0$, i.e. f is constant and we conclude. □

Remark 1.36. It is worth pointing out that the statement of Theorem 1.35 is similar in the minimal case: if $\varphi : M^m \to \mathbb{S}^{m+1}$ is a minimal hypersurface with $|A|^2 \leq m$, then either $|A| = 0$ or $|A|^2 = m$ (see [121]). Apparently, by way of contrast, an analog of Proposition 1.34 is not true in the minimal case. In fact, it was proved in [114] that if a minimal hypersurface $\varphi : M^3 \to \mathbb{S}^4$ has $|A|^2 > 3$, then $|A|^2 \geq 6$. But, if the compact minimal hypersurface of $\mathbb{S}^{m+1}$ with $|A|^2 \geq m$ has at most two distinct principal curvatures, then $|A|^2 = m$ (see [74]); and we believe that any proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ has at most two principal curvatures everywhere.

Obviously, from Proposition 1.34 and Theorem 1.35 we get the following result.

Proposition 1.37 ([16]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper-biharmonic and $|A|^2$ is constant, then φ is CMC and $|A|^2 = m$.*

The next result is a direct consequence of Proposition 1.34.

Proposition 1.38 ([16]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper-biharmonic and $|H|^2 \geq 4(m-1)/(m(m+8))$, then φ is CMC. Moreover,* *(i) if* $m \in \{2, 3\}$*, then* $\varphi(M)$ *is a small hypersphere* $\mathbb{S}^m(1/\sqrt{2})$*;*

(ii) if $m = 4$*, then* $\varphi(M)$ *is a small hypersphere* $\mathbb{S}^4(1/\sqrt{2})$ *or a standard product of spheres* $\mathbb{S}^3(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$*.*

Proof. Taking into account (1.84), the hypotheses imply $|A|^2 \geq m$. □

For the non-compact case we obtain the following.

Proposition 1.39 ([16])**.** *Let* $\varphi : M^m \to \mathbb{S}^{m+1}$*,* $m > 2$*, be a non-compact hypersurface. Assume that* M *is complete and has non-negative Ricci curvature. If* φ *is proper-biharmonic,* $|A|^2$ *is constant and* $|A|^2 \geq m$*, then* φ *is CMC and* $|A|^2 = m$*. In this case* $|H|^2 \leq ((m-2)/m)^2$*.*

Proof. We may assume that M is orientable (otherwise, we consider the double covering $\tilde{M}$ of M, which is non-compact, connected, complete, orientable, proper-biharmonic and with non-negative Ricci curvature; the final result will remain unchanged). We consider η to be a global unit section in the normal bundle NM of M in $\mathbb{S}^{m+1}$. Then, on M, we have

$$\Delta f = (m - |A|^2) f, \tag{1.87}$$

where $f = (\operatorname{trace} A)/m$, and

$$\frac{1}{2} \Delta f^2 = (m - |A|^2) f^2 - |\operatorname{grad} f|^2 \leq 0. \tag{1.88}$$

On the other hand, as $f^2 = |H|^2 \leq |A|^2/m$ is bounded, by the Omori-Yau Maximum Principle (see, for example, [136]), there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset M$ such that

$$\Delta f^2(p_k) > -\frac{1}{k} \qquad \text{and} \qquad \lim_{k \to \infty} f^2(p_k) = \sup_M f^2.$$

It follows that $\lim_{k \to \infty} \Delta f^2(p_k) = 0$, so $\lim_{k \to \infty} ((m - |A|^2) f^2(p_k)) = 0$.

As $\lim_{k \to \infty} f^2(p_k) = \sup_M f^2 > 0$, we get $|A|^2 = m$. But from (1.87) follows that f is a harmonic function on M. As f is also a bounded function on M, by a result of Yau (see [136]), we deduce that $f =$ constant. □

Corollary 1.40 ([16])**.** *Let* $\varphi : M^m \to \mathbb{S}^{m+1}$ *be a non-compact hypersurface. Assume that* M *is complete and has non-negative Ricci curvature. If* φ *is proper-biharmonic,* $|A|^2$ *is constant and* $|H|^2 \geq 4(m-1)/(m(m+8))$*, then* φ *is CMC and* $|A|^2 = m$*. In this case,* $m \geq 4$ *and* $|H|^2 \leq ((m-2)/m)^2$*.*

Proposition 1.41 ([16])**.** *Let* $\varphi : M^m \to \mathbb{S}^{m+1}$ *be a non-compact hypersurface. Assume that* M *is complete and has non-negative Ricci curvature. If* φ *is proper-biharmonic,* $|A|^2$ *is constant,* $|A|^2 \leq m$ *and* H *is nowhere zero, then* φ *is CMC and* $|A|^2 = m$*.*

Proof. As H is nowhere zero we consider $\eta = H/|H|$ a global unit section in the normal bundle. Then, on M,

$$\Delta f = (m - |A|^2) f, \tag{1.89}$$

where $f = |H| > 0$. As $m - |A|^2 \geq 0$ by a classical result (see, for example, [89, pag. 2]) we conclude that $m = |A|^2$ and therefore f is constant. □

1.4.3 Case 3

We first present another result of J.H. Chen in [48]. In order to do that, we shall need the following lemma.

Lemma 1.42. *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be an orientable hypersurface, η a unit section in the normal bundle, and put $A_\eta = A$. Then*

(i) $(\nabla A)(\cdot, \cdot)$ *is symmetric,*

(ii) $\langle (\nabla A)(\cdot, \cdot), \cdot \rangle$ *is totally symmetric,*

(iii) $\operatorname{trace}(\nabla A)(\cdot, \cdot) = m \operatorname{grad} f$.

Theorem 1.43 ([48])**.** *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact hypersurface. If φ is proper-biharmonic, M has non-negative sectional curvature and $m \leq 10$, then φ is CMC and $\varphi(M)$ is either $\mathbb{S}^m(1/\sqrt{2})$, or $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Proof. For the same reasons as in Theorem 1.35 we include a detailed proof of this result. We can assume that M is orientable (otherwise, as in the proof of Theorem 1.35, we work with the oriented double covering of M). Fix a unit section $\eta \in C(NM)$ and put $A = A_\eta$ and $f = (\operatorname{trace} A)/m$.

We intend to prove that the following inequality holds on M,

$$\frac{1}{2}\Delta\left(|A|^2 + \frac{m^2}{2}f^2\right) \leq \frac{3m^2(m-10)}{4(m-1)}|\operatorname{grad} f|^2 - \frac{1}{2}\sum_{i,j=1}^{m}(\lambda_i - \lambda_j)^2 R_{ijij}. \tag{1.90}$$

From the Weitzenböck formula we have

$$\frac{1}{2}\Delta|A|^2 = \langle \Delta A, A\rangle - |\nabla A|^2. \tag{1.91}$$

Let us first verify that

$$\operatorname{trace}(\nabla^2 A)(X, \cdot, \cdot) = \nabla_X(\operatorname{trace} \nabla A), \tag{1.92}$$

for all $X \in C(TM)$. Fix $p \in M$ and let $\{E_i\}_{i=1}^n$ be a local orthonormal frame field, geodesic at p. Then, also using Lemma 1.42(i), we get at p,

$$\begin{aligned}
\operatorname{trace}(\nabla^2 A)(X, \cdot, \cdot) &= \sum_{i=1}^{m}(\nabla^2 A)(X, E_i, E_i) = \sum_{i=1}^{m}(\nabla_X \nabla A)(E_i, E_i) \\
&= \sum_{i=1}^{m}\{\nabla_X \nabla A(E_i, E_i) - 2\nabla A(\nabla_X E_i, E_i)\} = \sum_{i=1}^{m}\nabla_X \nabla A(E_i, E_i) \\
&= \nabla_X(\operatorname{trace} \nabla A).
\end{aligned}$$

Using Lemma 1.42, the Ricci commutation formula (see, for example, [25]) and (1.92), we obtain

$$\begin{aligned}
\Delta A(X) &= -(\operatorname{trace} \nabla^2 A)(X) = -\operatorname{trace}(\nabla^2 A)(\cdot, \cdot, X) = -\operatorname{trace}(\nabla^2 A)(\cdot, X, \cdot) \\
&= -\operatorname{trace}(\nabla^2 A)(X, \cdot, \cdot) - \operatorname{trace}(RA)(\cdot, X, \cdot) \\
&= -\nabla_X(\operatorname{trace} \nabla A) - \operatorname{trace}(RA)(\cdot, X, \cdot) \\
&= -m\nabla_X \operatorname{grad} f - \operatorname{trace}(RA)(\cdot, X, \cdot),
\end{aligned} \tag{1.93}$$

where

$$RA(X,Y,Z) = R(X,Y)A(Z) - A(R(X,Y)Z), \quad \forall\, X,Y,Z \in C(TM).$$

Also, using (1.5)(ii) and Lemma 1.42, we obtain

$$\begin{aligned}
\operatorname{trace}\langle A(\nabla_{\cdot} \operatorname{grad} f), \cdot\rangle &= \operatorname{trace}\langle \nabla_{\cdot} A(\operatorname{grad} f) - (\nabla A)(\cdot, \operatorname{grad} f), \cdot\rangle \\
&= -\frac{m}{4} \operatorname{trace}\langle \nabla_{\cdot} \operatorname{grad} f^2, \cdot\rangle - \langle \operatorname{trace}(\nabla A), \operatorname{grad} f\rangle \\
&= \frac{m}{4}\Delta f^2 - m|\operatorname{grad} f|^2. \qquad (1.94)
\end{aligned}$$

Using (1.93) and (1.94), we get

$$\begin{aligned}
\langle \Delta A, A\rangle &= \operatorname{trace}\langle \Delta A(\cdot), A(\cdot)\rangle \\
&= -m \operatorname{trace}\langle \nabla_{\cdot} \operatorname{grad} f, A(\cdot)\rangle + \langle T, A\rangle \\
&= -m \operatorname{trace}\langle A(\nabla_{\cdot} \operatorname{grad} f), \cdot\rangle + \langle T, A\rangle \\
&= m^2|\operatorname{grad} f|^2 - \frac{m^2}{4}\Delta f^2 + \langle T, A\rangle, \qquad (1.95)
\end{aligned}$$

where $T(X) = -\operatorname{trace}(RA)(\cdot, X, \cdot)$, $X \in C(TM)$.

In the following we shall verify that

$$|\nabla A|^2 \geq \frac{m^2(m+26)}{4(m-1)}|\operatorname{grad} f|^2, \qquad (1.96)$$

at every point of M. Now, let us fix a point $p \in M$.

If $\operatorname{grad}_p f = 0$, then (1.96) obviously holds at p.

If $\operatorname{grad}_p f \neq 0$, then on a neighborhood $U \subset M$ of p we can consider an orthonormal frame field $E_1 = (\operatorname{grad} f)/|\operatorname{grad} f|$, $E_2, \ldots, E_m$, where $E_k(f) = 0$, for all $k = 2, \ldots, m$. Using (1.5)(ii), we obtain on U

$$\begin{aligned}
\langle (\nabla A)(E_1, E_1), E_1\rangle &= \frac{1}{|\operatorname{grad} f|^3}\big(\langle \nabla_{\operatorname{grad} f} A(\operatorname{grad} f), \operatorname{grad} f\rangle \\
&\quad - \langle A(\nabla_{\operatorname{grad} f} \operatorname{grad} f), \operatorname{grad} f\rangle\big) \\
&= -\frac{m}{2}|\operatorname{grad} f|. \qquad (1.97)
\end{aligned}$$

From here, using Lemma 1.42, we also have on U

$$\begin{aligned}
\sum_{k=2}^{m}\langle (\nabla A)(E_k, E_k), E_1\rangle &= \sum_{i=1}^{m}\langle (\nabla A)(E_i, E_i), E_1\rangle - \langle (\nabla A)(E_1, E_1), E_1\rangle \\
&= \langle \operatorname{trace} \nabla A, E_1\rangle + \frac{m}{2}|\operatorname{grad} f| = \frac{3m}{2}|\operatorname{grad} f|. \qquad (1.98)
\end{aligned}$$

Using (1.97) and (1.98), we have on U

$$\begin{aligned}
|\nabla A|^2 &= \sum_{i,j=1}^{m} |(\nabla A)(E_i, E_j)|^2 = \sum_{i,j,h=1}^{m} \langle(\nabla A)(E_i, E_j), E_h\rangle^2 \\
&\geq \langle(\nabla A)(E_1, E_1), E_1\rangle^2 + 3\sum_{k=2}^{m}\langle(\nabla A)(E_k, E_k), E_1\rangle^2 \\
&\geq \langle(\nabla A)(E_1, E_1), E_1\rangle^2 + \frac{3}{m-1}\left(\sum_{k=2}^{m}\langle(\nabla A)(E_k, E_k), E_1\rangle\right)^2 \\
&= \frac{m^2(m+26)}{4(m-1)}|\operatorname{grad} f|^2, \qquad (1.99)
\end{aligned}$$

thus (1.96) is verified, and (1.91) implies

$$\frac{1}{2}\Delta\left(|A|^2 + \frac{m^2}{2}f^2\right) \leq \frac{3m^2(m-10)}{4(m-1)}|\operatorname{grad} f|^2 + \langle T, A\rangle. \qquad (1.100)$$

Fix $p \in M$ and consider $\{e_i\}_{i=1}^m$ to be an orthonormal basis of T_pM, such that $A(e_i) = \lambda_i e_i$. Then, at p, we get

$$\langle T, A\rangle = -\frac{1}{2}\sum_{i,j=1}^{m}(\lambda_i - \lambda_j)^2 R_{ijij},$$

and then (1.100) becomes (1.90).

Now, since $m \leq 10$ and M has non-negative sectional curvature, we obtain

$$\Delta\left(|A|^2 + \frac{m^2}{2}|H|^2\right) \leq 0$$

on M. As M is compact, we have

$$\Delta\left(|A|^2 + \frac{m^2}{2}|H|^2\right) = 0$$

on M, which implies

$$(\lambda_i - \lambda_j)^2 R_{ijij} = 0 \qquad (1.101)$$

on M. Fix $p \in M$. From the Gauss equation for φ, $R_{ijij} = 1 + \lambda_i\lambda_j$, for all $i \neq j$, and from (1.101) we obtain

$$(\lambda_i - \lambda_j)(1 + \lambda_i\lambda_j) = 0, \quad i \neq j.$$

Let us now fix λ_1. If there exists another principal curvature $\lambda_j \neq \lambda_1$, $j > 1$, then from the latter relation we get that $\lambda_1 \neq 0$ and $\lambda_j = -1/\lambda_1$. Thus φ has at most two distinct principal curvatures at p. Since p was arbitrarily fixed, we obtain that φ has at most two distinct principal curvatures everywhere and we conclude by using Theorem 1.21. □

Proposition 1.44 ([16]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$, $m \geq 3$, be a hypersurface. Assume that M has non-negative sectional curvature and for all $p \in M$ there exists $X_p \in T_pM$, $|X_p| = 1$, such that* $\mathrm{Ricci}(X_p, X_p) = 0$. *If φ is proper-biharmonic, then $\varphi(M)$ is an open part of $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Proof. Let $p \in M$ be an arbitrarily fixed point, and $\{e_i\}_{i=1}^m$ an orthonormal basis in T_pM such that $A(e_i) = \lambda_i e_i$. For $i \neq j$, using (1.80), we have that $\mathrm{Ricci}(e_i, e_j) = 0$. Therefore, $\{e_i\}_{i=1}^m$ is also a basis of eigenvectors for the Ricci curvature. Now, if $\mathrm{Ricci}(e_i, e_i) > 0$ for all $i = 1, \dots m$, then $\mathrm{Ricci}(X, X) > 0$ for all $X \in T_pM \setminus \{0\}$. Thus there must exist i_0 such that $\mathrm{Ricci}(e_{i_0}, e_{i_0}) = 0$. Assume that $\mathrm{Ricci}(e_1, e_1) = 0$. From

$$0 = \mathrm{Ricci}(e_1, e_1) = \sum_{j=2}^{m} R_{1j1j} = \sum_{j=2}^{m} K_{1j}$$

and since $K_{1j} \geq 0$ for all $j \geq 2$, we conclude that $K_{1j} = 0$ for all $j \geq 2$, that is $1 + \lambda_1 \lambda_j = 0$ for all $j \geq 2$. The latter implies that $\lambda_1 \neq 0$ and $\lambda_j = -1/\lambda_1$ for all $j \geq 2$. Thus M has two distinct principal curvatures everywhere, one of them of multiplicity one. □

Remark 1.45. If $\varphi : M^m \to \mathbb{S}^{m+1}$, $m \geq 3$, is a compact hypersurface, then the conclusion of Proposition 1.44 holds replacing the hypothesis on the Ricci curvature with the requirement that the first fundamental group is infinite. In fact, the full classification of compact hypersurfaces in $\mathbb{S}^{m+1}$ with non-negative sectional curvature and infinite first fundamental group was given in [50].

By imposing conditions on the scalar curvature, other classification results can be obtained. We first have the following estimate for the scalar curvature of compact proper-biharmonic hypersurfaces with constant scalar curvature in spheres.

Proposition 1.46. *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a compact proper-biharmonic hypersurface with constant scalar curvature s. Then*

$$m(m-2) < s \leq 2m(m-1).$$

Moreover, $s = 2m(m-1)$ if and only if $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$.

Proof. From the Gauss equation one gets

$$s = m(m-1) + m^2 f^2 - |A|^2, \tag{1.102}$$

and, together with (1.75), this implies

$$\frac{1}{2}\Delta f^2 = (s - m(m-2))f^2 - m^2 f^4 - |\operatorname{grad} f|^2. \tag{1.103}$$

Suppose that $s \leq m(m-2)$. Then, from (1.103) we obtain $\Delta f^2 \leq 0$, and since M is compact this implies that $f^2 = \text{constant}$. Now, (1.103) implies $f^2 = 0$ and we have a contradiction.

Suppose that $s \geq 2m(m-1)$. Then, from (1.102), since $|A|^2 \geq mf^2$, we obtain $f^2 \geq 1$. Using Corollary 3.3 in [23], we get $f^2 = 1$, thus $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ and $s = 2m(m-1)$.

□

Now, using Theorem 2 in [88], we obtain the following classification.

Theorem 1.47. *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a proper-biharmonic compact hypersurface with constant scalar curvature s, $m \geq 3$. If*

(i) $s \geq m(m-1)$,

(ii) the squared norm of the shape operator $|A|^2$ satisfies

$$\frac{s - m(m-1)}{m-1} \leq |A|^2 \leq \frac{s-(m-2)(m-1)}{m-2} + \frac{(m-2)(m-1)}{s-(m-2)(m-1)},$$

then $|A|^2 = m$ and either $s = 2m(m-1)$ and $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$, or $s = 2(m-2)(m-1)$ and $\varphi(M) = \mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^{m-1}(1/\sqrt{2})$.

By Theorem 11 in [137] and Theorem 2 in [49], with a restriction on the sectional curvature, we obtain the next rigidity result.

Theorem 1.48. *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a proper-biharmonic compact hypersurface. Suppose M has constant scalar curvature s and non-negative sectional curvature.*

(i) If M has positive sectional curvature, then $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$.

(ii) If $s \geq m(m-1)$, then either $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ or $\varphi(M) = \mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

1.5 PMC biharmonic immersions in $\mathbb{S}^n$

In this section we present some of the most important results on PMC biharmonic submanifolds in spheres. In order to do that we first need the following lemma.

Lemma 1.49 ([16])**.** *Let $\varphi : M^m \to N^n$ be an immersion. Then $|A_H|^2 \leq |H|^2|B|^2$ on M. Moreover, $|A_H|^2 = |H|^2|B|^2$ at $p \in M$ if and only if either $H(p) = 0$, or the first normal of φ at p is spanned by $H(p)$.*

Proof. Let $p \in M$. If $|H(p)| = 0$, then the conclusion is obvious. Consider now the case when $|H(p)| \neq 0$, let $\eta_p = H(p)/|H(p)| \in N_pM$ and let $\{e_i\}_{i=1}^m$ be a basis in T_pM. Then, at p,

$$\begin{aligned} |A_H|^2 &= \sum_{i,j=1}^m \langle A_H(e_i), e_j\rangle^2 = \sum_{i,j=1}^m \langle B(e_i,e_j), H\rangle^2 = |H|^2 \sum_{i,j=1}^m \langle B(e_i,e_j), \eta_p\rangle^2 \\ &\leq |H|^2|B|^2. \end{aligned}$$

In this case equality holds if and only if $\sum_{i,j=1}^m \langle B(e_i,e_j), \eta_p\rangle^2 = |B|^2$, i.e.

$$\langle B(e_i,e_j), \xi_p\rangle = 0, \quad \forall\, \xi_p \in N_pM \text{ with } \xi_p \perp H(p).$$

This is equivalent to the first normal at p being spanned by $H(p)$ and we conclude. $\square$

Using the above lemma we can prove the following lower bound for the norm of the second fundamental form.

Proposition 1.50 ([16]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper-biharmonic immersion. Then $m \leq |B|^2$ and equality holds if and only if φ induces a CMC proper-biharmonic immersion of M into a totally geodesic sphere $\mathbb{S}^{m+1} \subset \mathbb{S}^n$.*

Proof. By Corollary 1.3 we have $|A_H|^2 = m|H|^2$ and, by using Lemma 1.49, we obtain $m \leq |B|^2$.

Since H is parallel and nowhere zero, equality holds if and only if the first normal is spanned by H, and we can apply the codimension reduction result of J. Erbacher ([62]) to obtain the existence of a totally geodesic sphere $\mathbb{S}^{m+1} \subset \mathbb{S}^n$, such that φ is an immersion of M into $\mathbb{S}^{m+1}$. Since $\varphi : M^m \to \mathbb{S}^n$ is PMC proper-biharmonic, the restriction $M^m \to \mathbb{S}^{m+1}$ is CMC proper-biharmonic. □

Remark 1.51.
(i) Let $\varphi = \imath \circ \phi : M \to \mathbb{S}^n$ be a proper-biharmonic immersion of class **B3**. Then $m \leq |B|^2$ and equality holds if and only if the induced ϕ is totally geodesic.

(ii) Let $\varphi = \imath \circ (\phi_1 \times \phi_2) : M_1 \times M_2 \to \mathbb{S}^n$ be a proper-biharmonic immersion of class **B4**. Then $m \leq |B|^2$ and equality holds if and only if both ϕ_1 and ϕ_2 are totally geodesic.

The above remark suggests to look for PMC proper-biharmonic immersions with $|H| = 1$ and $|B|^2 = m$.

Corollary 1.52 ([16]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper-biharmonic immersion. Then $|H| = 1$ and $|B|^2 = m$ if and only if $\varphi(M)$ is an open part of $\mathbb{S}^m(1/\sqrt{2}) \subset \mathbb{S}^{m+1} \subset \mathbb{S}^n$.*

The case when M is a surface is more rigid. Using the classification of PMC surfaces in $\mathbb{S}^n$ given by S.-T. Yau [138], and [21, Corollary 5.5], we obtain the following result.

Theorem 1.53 ([21]). *Let $\varphi : M^2 \to \mathbb{S}^n$ be a PMC proper-biharmonic surface. Then φ induces a minimal immersion of M into a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$.*

If $n = 4$ in Theorem 1.53, then the same conclusion holds under the weakened assumption that the surface is CMC. In order to prove this, the following result is also needed.

Theorem 1.54 ([21]). *Let $\varphi : M^m \to \mathbb{S}^{m+2}$ be a pseudo-umbilical submanifold, $m \neq 4$. Then M is proper-biharmonic in $\mathbb{S}^{m+2}$ if and only if it is minimal in $\mathbb{S}^{m+1}(1/\sqrt{2})$.*

We have

Theorem 1.55 ([24]). *Let $\varphi : M^2 \to \mathbb{S}^4$ be a CMC proper-biharmonic surface in $\mathbb{S}^4$. Then M^2 is minimal in $\mathbb{S}^3(1/\sqrt{2})$.*

Proof. Following [46], we shall first prove that any proper-biharmonic CMC surface in $\mathbb{S}^4$ is PMC. Then we shall conclude by using Theorem 1.53.

Denote by H the mean curvature vector field of M^2 in $\mathbb{S}^4$. Since M is CMC proper-biharmonic with constant mean curvature, its mean curvature does not vanish at any point and we denote by

$$E_3 = \frac{H}{|H|} \in C(NM). \tag{1.104}$$

Consider $\{E_1, E_2\}$ to be a local orthonormal frame field on M around an arbitrary fixed point $p \in M$ and let E_4 be a local unit section in the normal bundle, orthogonal to E_3. We can assume that $\{E_1, E_2, E_3, E_4\}$ is the restriction of a local orthonormal frame field around p on $\mathbb{S}^4$, also denoted by $\{E_1, E_2, E_3, E_4\}$.

Denote by B the second fundamental form of M in $\mathbb{S}^4$ and by A_3 and A_4 the Weingarten operators associated to E_3 and E_4, respectively.

Let $\nabla^{\mathbb{S}^4}$ and ∇ be the Levi-Civita connections on $\mathbb{S}^4$ and on M, respectively, and denote by ω_A^B the connection 1-forms of $\mathbb{S}^4$ with respect to $\{E_1, E_2, E_3, E_4\}$, i.e.

$$\nabla^{\mathbb{S}^4} E_A = \omega_A^B E_B, \qquad A, B = 1, \dots, 4. \tag{1.105}$$

From (1.104) we have $H = |H| E_3$ and, since $2H = B(E_1, E_1) + B(E_2, E_2)$, we obtain that

$$\begin{aligned} 0 &= 2\langle H, E_4\rangle = \langle B(E_1, E_1), E_4\rangle + \langle B(E_2, E_2), E_4\rangle \\ &= \langle A_4(E_1), E_1\rangle + \langle A_4(E_2), E_2\rangle, \end{aligned} \tag{1.106}$$

i.e. $\operatorname{trace} A_4 = 0$. As a consequence, we have

$$\begin{aligned} |A_4|^2 &= |A_4(E_1)|^2 + |A_4(E_2)|^2 \\ &= \langle A_4(E_1), E_1\rangle^2 + 2\langle A_4(E_1), E_2\rangle^2 + \langle A_4(E_2), E_2\rangle^2 \\ &= 2\big(\langle A_4(E_1), E_1\rangle^2 + \langle A_4(E_1), E_2\rangle^2\big). \end{aligned} \tag{1.107}$$

The tangent part of the biharmonic equation (1.2) now writes

$$A_{\nabla^\perp_{E_1} E_3}(E_1) + A_{\nabla^\perp_{E_2} E_3}(E_2) = 0. \tag{1.108}$$

Since

$$\begin{aligned} \nabla^\perp_{E_1} E_3 &= \langle \nabla^\perp_{E_1} E_3, E_3\rangle E_3 + \langle \nabla^\perp_{E_1} E_3, E_4\rangle E_4 = \langle \nabla^{\mathbb{S}^4}_{E_1} E_3, E_4\rangle E_4 \\ &= \omega_3^4(E_1) E_4, \end{aligned}$$

and

$$\nabla^\perp_{E_2} E_3 = \omega_3^4(E_2) E_4,$$

from (1.108) we get

$$\omega_3^4(E_1) A_4(E_1) + \omega_3^4(E_2) A_4(E_2) = 0. \tag{1.109}$$

Considering now the scalar product by E_1 and E_2 in (1.109), we obtain

$$\begin{cases} \langle A_4(E_1), E_1\rangle \omega_3^4(E_1) + \langle A_4(E_2), E_1\rangle \omega_3^4(E_2) = 0, \\ \\ \langle A_4(E_1), E_2\rangle \omega_3^4(E_1) + \langle A_4(E_2), E_2\rangle \omega_3^4(E_2) = 0. \end{cases} \tag{1.110}$$

Equations (1.110) can be thought of as a linear homogeneous system in $\omega_3^4(E_1)$ and $\omega_3^4(E_2)$. By using (1.106) and (1.107), the determinant of this system is equal to $-\frac{1}{2}|A_4|^2$.

Suppose now that $(\nabla^\perp H)(p) \neq 0$. Then there exists a neighborhood U of p in M such that $\nabla^\perp H \neq 0$, at any point of U. Since

$$\nabla^\perp H = |H|\nabla^\perp E_3 = |H|\{\omega_3^4(E_1)E_1^\flat \otimes E_4 + \omega_3^4(E_2)E_2^\flat \otimes E_4\},$$

the hypothesis $\nabla^\perp H \neq 0$ on U implies that (1.110) admits non-trivial solutions at any point of U. Therefore, the determinant of (1.110) is zero, which means that $|A_4|^2 = 0$, i.e. $A_4 = 0$ on U.

We have two cases.

Case I. If U is pseudo-umbilical in $\mathbb{S}^4$, i.e. $A_3 = |H|\,\mathrm{Id}$, from Theorem 1.54 we get that U is minimal in $\mathbb{S}^3(1/\sqrt{2})$ and we have a contradiction, since any minimal surface in $\mathbb{S}^3(1/\sqrt{2})$ has parallel mean curvature vector field in $\mathbb{S}^4$.

Case II. Suppose that there exists $q \in U$ such that $A_3(q) \neq |H|\,\mathrm{Id}$. Then, eventually by restricting U, we can suppose that $A_3 \neq |H|\,\mathrm{Id}$ on U. Since the principal curvatures of A_3 have constant multiplicity 1, we can suppose that E_1 and E_2 are such that

$$A_3(E_1) = k_1E_1, \qquad A_3(E_2) = k_2E_2,$$

where $k_1 \neq k_2$ at any point of U. As $A_4 = 0$, we obtain

$$B(E_1,E_1) = k_1E_3, \quad B(E_1,E_2) = 0, \quad B(E_2,E_2) = k_2E_3, \tag{1.111}$$

on U.

In the following we shall use the Codazzi and Gauss equations in order to get to a contradiction.

The Codazzi equation is given in this setting by

$$0 = (\nabla_X^{\mathbb{S}^4}B)(Y,Z,\eta) - (\nabla_Y^{\mathbb{S}^4}B)(X,Z,\eta), \quad \forall X,Y,Z \in C(TM), \forall \eta \in C(NM), \tag{1.112}$$

where $\nabla_X^{\mathbb{S}^4}B$ is defined by

$$\begin{aligned}(\nabla_X^{\mathbb{S}^4}B)(Y,Z,\eta) &= X\langle B(Y,Z),\eta\rangle - \langle B(\nabla_XY,Z),\eta\rangle - \langle B(Y,\nabla_XZ),\eta\rangle \\ &\quad -\langle B(Y,Z),\nabla_X^\perp\eta\rangle.\end{aligned}$$

For $X = Z = E_1$, $Y = E_2$ and $\eta = E_3$, equation (1.112) leads to

$$\begin{aligned}0 &= E_1\langle B(E_2,E_1),E_3\rangle - E_2\langle B(E_1,E_1),E_3\rangle \\ &\quad -\langle B(\nabla_{E_1}E_2,E_1),E_3\rangle + \langle B(\nabla_{E_2}E_1,E_1),E_3\rangle \\ &\quad -\langle B(E_2,\nabla_{E_1}E_1),E_3\rangle + \langle B(E_1,\nabla_{E_2}E_1),E_3) \\ &\quad -\langle B(E_2,E_1),\nabla_{E_1}^\perp E_3\rangle + \langle B(E_1,E_1),\nabla_{E_2}^\perp E_3\rangle.\end{aligned} \tag{1.113}$$

Now, from (1.111) we have

$$B(\nabla_{E_1}E_2,E_1) = k_1\omega_2^1(E_1)E_3, \qquad B(E_2,\nabla_{E_1}E_1) = -k_2\omega_2^1(E_1)E_3,$$

$$B(\nabla_{E_2}E_1, E_1) = 0, \qquad \langle B(E_1, E_1), \nabla^{\perp}_{E_2}E_3\rangle = 0,$$

thus (1.113) implies

$$E_2(k_1) = (k_2 - k_1)\omega_2^1(E_1). \tag{1.114}$$

Analogously, for $X = Z = E_2$, $Y = E_1$ and $\eta = E_3$ in (1.112), we obtain

$$E_1(k_2) = (k_2 - k_1)\omega_2^1(E_2). \tag{1.115}$$

For $X = Z = E_1$, $Y = E_2$ and $\eta = E_4$ in (1.112), we obtain

$$\begin{aligned} 0 &= \langle B(E_2, E_1), \nabla^{\perp}_{E_1}E_4\rangle - \langle B(E_1, E_1), \nabla^{\perp}_{E_2}E_4\rangle \\ &= -k_1\langle E_3, \nabla^{\perp}_{E_2}E_4\rangle, \end{aligned}$$

which implies

$$k_1\omega_3^4(E_2) = 0. \tag{1.116}$$

Analogously, for $X = Z = E_2$, $Y = E_1$ and $\eta = E_4$ in (1.112), we obtain

$$k_2\omega_3^4(E_1) = 0. \tag{1.117}$$

Since $\nabla^{\perp}H \neq 0$ on U, we can suppose that $\omega_3^4(E_1) \neq 0$ on U. This, together with (1.117), leads to $k_2 = 0$. From here we get $|k_1| = 2|H| \neq 0$, and consequently k_1 is a non-zero constant. As $k_1 \neq k_2$, from (1.114) and (1.115) we obtain

$$\omega_2^1(E_1) = \omega_2^1(E_2) = 0, \tag{1.118}$$

thus M is flat.
Consider now the Gauss equation (1.50). As M is flat, for $X = W = E_1$ and $Y = Z = E_2$, equations (1.50) and (1.111) lead to

$$\begin{aligned} 1 &= \langle B(E_1, E_2), B(E_2, E_1)\rangle - \langle B(E_1, E_1), B(E_2, E_2)\rangle = -k_1k_2 \\ &= 0, \end{aligned} \tag{1.119}$$

and we have a contradiction.

Therefore, $\nabla^{\perp}H = 0$ and we conclude. □

For the higher dimensional case, in [23] there were obtained bounds for the value of the mean curvature of a PMC proper-biharmonic immersion.

We shall further see that, when $m > 2$, the situation is more complex and, apart from 1, the mean curvature can assume other lower values, as expected in view of Theorem 1.6.

First, let us prove an auxiliary result, concerning non-full proper biharmonic submanifolds of $\mathbb{S}^n$, which generalizes Theorem 5.4 in [21].

Proposition 1.56 ([23]). *Let $\psi : M^m \to \mathbb{S}^{n-1}(a)$ be a submanifold of a small hypersphere $\mathbb{S}^{n-1}(a)$ in $\mathbb{S}^n$, $a \in (0, 1)$. Then M is proper-biharmonic in $\mathbb{S}^n$ if and only if either $a = 1/\sqrt{2}$ and M is minimal in $\mathbb{S}^{n-1}(1/\sqrt{2})$, or $a > 1/\sqrt{2}$ and M is minimal in a small hypersphere $\mathbb{S}^{n-2}(1/\sqrt{2})$ in $\mathbb{S}^{n-1}(a)$. In both cases, $|H| = 1$.*

Proof. The converse follows immediately by using Theorem 1.5.

In order to prove the other implication, denote by $\mathbf{j}$ and $\mathbf{i}$ the inclusion maps of M in $\mathbb{S}^{n-1}(a)$ and of $\mathbb{S}^{n-1}(a)$ in $\mathbb{S}^n$, respectively.

Up to an isometry of $\mathbb{S}^n$, we can consider

$$\mathbb{S}^{n-1}(a) = \left\{ (x^1, \ldots, x^n, \sqrt{1-a^2}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} (x^i)^2 = a^2 \right\} \subset \mathbb{S}^n.$$

Then

$$C(T\mathbb{S}^{n-1}(a)) = \left\{ (X^1, \ldots, X^n, 0) \in C(T\mathbb{R}^{n+1}) : \sum_{i=1}^{n} x^i X^i = 0 \right\},$$

while $\eta = \frac{1}{c}\left(x^1, \ldots, x^n, -\frac{a^2}{\sqrt{1-a^2}}\right)$ is a unit section in the normal bundle of $\mathbb{S}^{n-1}(a)$ in $\mathbb{S}^n$, where $c^2 = \frac{a^2}{1-a^2}$, $c > 0$. The tension and bitension fields of the inclusion $\imath = \mathbf{i} \circ \mathbf{j} : M \to \mathbb{S}^n$, are given by

$$\tau(\imath) = \tau(\mathbf{j}) - \frac{m}{c}\eta, \qquad \tau_2(\imath) = \tau_2(\mathbf{j}) - \frac{2m}{c^2}\tau(\mathbf{j}) + \frac{1}{c}\left\{ |\tau(\mathbf{j})|^2 - \frac{m^2}{c^2}(c^2 - 1) \right\} \eta.$$

Since M is biharmonic in $\mathbb{S}^n$, we obtain

$$\tau_2(\mathbf{j}) = \frac{2m}{c^2}\tau(\mathbf{j}) \tag{1.120}$$

and

$$|\tau(\mathbf{j})|^2 = \frac{m^2}{c^2}(c^2 - 1) = \frac{m^2}{a^2}(2a^2 - 1).$$

From here $a \geq 1/\sqrt{2}$. Also,

$$|\tau(\imath)|^2 = |\tau(\mathbf{j})|^2 + \frac{m^2}{c^2} = m^2.$$

This implies that the mean curvature of M in $\mathbb{S}^n$ is 1.

The case $a = 1/\sqrt{2}$ is solved by Theorem 1.5.

Consider $a > 1/\sqrt{2}$, thus $\tau(\mathbf{j}) \neq 0$. As $|H| = 1$, by applying Theorem 1.7, M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-1}(1/\sqrt{2}) \subset \mathbb{S}^n$, so it is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{S}^n$ ([45]). From here it can be proved that M is also pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{S}^{n-1}(a)$. As M is not minimal in $\mathbb{S}^{n-1}(a)$, it follows that M is a minimal submanifold of a small hypersphere $\mathbb{S}^{n-2}(b)$ in $\mathbb{S}^{n-1}(a)$. By a straightforward computation, equation (1.120) implies $b = 1/\sqrt{2}$ and the proof is completed.

□

Since every small sphere $\mathbb{S}^{n'}(a)$ in $\mathbb{S}^n$, $a \in (0,1)$, is contained into a great sphere $\mathbb{S}^{n'+1}$ of $\mathbb{S}^n$, from Proposition 1.56 we have the following.

Corollary 1.57 ([23]). *Let $\psi : M^m \to \mathbb{S}^{n'}(a)$ be a submanifold of a small sphere $\mathbb{S}^{n'}(a)$ in $\mathbb{S}^n$, $a \in (0,1)$. Then M is proper-biharmonic in $\mathbb{S}^n$ if and only if either $a = 1/\sqrt{2}$ and M is minimal in $\mathbb{S}^{n'}(1/\sqrt{2})$, or $a > 1/\sqrt{2}$ and M is minimal in a small hypersphere $\mathbb{S}^{n'-1}(1/\sqrt{2})$ in $\mathbb{S}^{n'}(a)$. In both cases, $|H| = 1$.*

Let $\varphi : M^m \to \mathbb{S}^n$ be a submanifold in $\mathbb{S}^n$. For our purpose it is convenient to define, following [2] and [3], the $(1,1)$-tensor field $\Phi = A_H - |H|^2 I$, where I is the identity on $C(TM)$. We notice that Φ is symmetric, $\operatorname{trace}\Phi = 0$ and

$$|\Phi|^2 = |A_H|^2 - m|H|^4. \tag{1.121}$$

Moreover, $\Phi = 0$ if and only if M is pseudo-umbilical.

By using the Gauss equation of M in $\mathbb{S}^n$, one gets the curvature tensor field of M in terms of Φ as follows.

Lemma 1.58 ([23]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a submanifold with nowhere zero mean curvature vector field. Then the curvature tensor field of M is given by*

$$\begin{aligned} R(X,Y)Z &= (1+|H|^2)(\langle Z,Y\rangle X - \langle Z,X\rangle Y) \\ &\quad + \frac{1}{|H|^2}(\langle Z,\Phi(Y)\rangle \Phi(X) - \langle Z,\Phi(X)\rangle \Phi(Y)) \\ &\quad + \langle Z,\Phi(Y)\rangle X - \langle Z,\Phi(X)\rangle Y + \langle Z,Y\rangle \Phi(X) - \langle Z,X\rangle \Phi(Y) \\ &\quad + \sum_{a=1}^{k-1}\{\langle Z,A_{\eta_a}(Y)\rangle A_{\eta_a}(X) - \langle Z,A_{\eta_a}(X)\rangle A_{\eta_a}(Y)\}, \end{aligned} \tag{1.122}$$

for all $X,Y,Z \in C(TM)$, where $\{H/|H|,\eta_a\}_{a=1}^{k-1}$, $k = n-m$, denotes a local orthonormal frame field in the normal bundle of M in $\mathbb{S}^n$.

In the case of hypersurfaces, i.e. $k=1$, the previous result holds by making the convention that $\sum_{a=1}^{k-1}\{\ldots\} = 0$.

For what concerns the expression of $\operatorname{trace}\nabla^2\Phi$, which will be needed further, the following result holds.

Lemma 1.59 ([23]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a submanifold with nowhere zero mean curvature vector field. If $\nabla^\perp H = 0$, then $\nabla\Phi$ is symmetric and*

$$\begin{aligned} (\operatorname{trace}\nabla^2\Phi)(X) &= -|\Phi|^2 X + \left(m + m|H|^2 - \frac{|\Phi|^2}{|H|^2}\right)\Phi(X) + m\Phi^2(X) \\ &\quad - \sum_{a=1}^{k-1}\langle \Phi, A_{\eta_a}\rangle A_{\eta_a}(X). \end{aligned} \tag{1.123}$$

Proof. From the Codazzi equation, as $\nabla^\perp H = 0$, we get $(\nabla A_H)(X,Y) = (\nabla A_H)(Y,X)$, for all $X,Y \in C(TM)$, where

$$(\nabla A_H)(X,Y) = (\nabla_X A_H)(Y) = \nabla_X A_H(Y) - A_H(\nabla_X Y).$$

As the mean curvature of M is constant we have $\nabla\Phi = \nabla A_H$, thus $\nabla\Phi$ is symmetric; and $\operatorname{trace}(\nabla\Phi) = \operatorname{trace}(\nabla A_H) = 0$.

We recall the Ricci commutation formula

$$(\nabla^2\Phi)(X,Y,Z) - (\nabla^2\Phi)(Y,X,Z) = R(X,Y)\Phi(Z) - \Phi(R(X,Y)Z), \tag{1.124}$$

for all $X,Y,Z \in C(TM)$, where

$$\begin{aligned}(\nabla^2\Phi)(X,Y,Z) &= (\nabla_X\nabla\Phi)(Y,Z)\\ &= \nabla_X((\nabla\Phi)(Y,Z)) - (\nabla\Phi)(\nabla_XY,Z) - (\nabla\Phi)(Y,\nabla_XZ).\end{aligned}$$

Consider $\{X_i\}_{i=1}^m$ to be a local orthonormal frame field on M and $\{H/|H|,\eta_a\}_{a=1}^{k-1}$, $k=n-m$, a local orthonormal frame field in the normal bundle of M in $\mathbb{S}^n$. As η_a is orthogonal to H, we get $\operatorname{trace} A_{\eta_a}=0$, for all $a=1,\dots,k-1$. Using also the symmetry of Φ and $\nabla\Phi$, (1.124) and (1.122), we have

$$\begin{aligned}(\operatorname{trace}\nabla^2\Phi)(X) &= \sum_{i=1}^m(\nabla^2\Phi)(X_i,X_i,X) = \sum_{i=1}^m(\nabla^2\Phi)(X_i,X,X_i)\\ &= \sum_{i=1}^m\{(\nabla^2\Phi)(X,X_i,X_i) + R(X_i,X)\Phi(X_i) - \Phi(R(X_i,X)X_i)\}\\ &= \sum_{i=1}^m(\nabla^2\Phi)(X,X_i,X_i)\\ &\quad -|\Phi|^2X + \left(m+m|H|^2-\frac{|\Phi|^2}{|H|^2}\right)\Phi(X) + m\Phi^2(X)\\ &\quad +\sum_{a=1}^{k-1}\{(A_{\eta_a}\circ\Phi-\Phi\circ A_{\eta_a})(A_{\eta_a}(X)) - \langle\Phi,A_{\eta_a}\rangle A_{\eta_a}(X)\}.\end{aligned}$$

By a straightforward computation,

$$\sum_{i=1}^m(\nabla^2\Phi)(X,X_i,X_i) = \nabla_X(\operatorname{trace}\nabla\Phi) = 0.$$

Moreover, from the Ricci equation, since $\nabla^\perp H=0$, we obtain $A_{\eta_a}\circ A_H = A_H\circ A_{\eta_a}$, thus $A_{\eta_a}\circ\Phi = \Phi\circ A_{\eta_a}$, and we end the proof of this lemma.

□

We shall also use the following lemma.

Lemma 1.60 ([23])**.** *Let $\varphi : M^m \to \mathbb{S}^n$ be a submanifold with nowhere zero mean curvature vector field. If $\nabla^\perp H=0$ and A_H is orthogonal to A_{η_a}, for all $a=1,\dots,k-1$, then*

$$-\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + \left(m+m|H|^2-\frac{|\Phi|^2}{|H|^2}\right)|\Phi|^2 + m\operatorname{trace}\Phi^3. \qquad (1.125)$$

Proof. Since A_H is orthogonal to A_{η_a} and $\operatorname{trace}A_{\eta_a}=0$, we get $\langle\Phi,A_{\eta_a}\rangle=0$, for all $a=1,\dots,k-1$, and (1.123) becomes

$$(\operatorname{trace}\nabla^2\Phi)(X) = -|\Phi|^2X + \left(m+m|H|^2-\frac{|\Phi|^2}{|H|^2}\right)\Phi(X) + m\Phi^2(X). \qquad (1.126)$$

Now, the Weitzenböck formula,

$$-\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + \langle\Phi,\operatorname{trace}\nabla^2\Phi\rangle,$$

together with the symmetry of Φ and (1.126), leads to the conclusion. □

We also recall here the Okumura Lemma.

Lemma 1.61 (Okumura Lemma, [107]). *Let $b_1, \ldots, b_m$ be real numbers such that $\sum_{i=1}^{m} b_i = 0$. Then*

$$-\frac{m-2}{\sqrt{m(m-1)}}\left(\sum_{i=1}^{m} b_i^2\right)^{3/2} \leq \sum_{i=1}^{m} b_i^3 \leq \frac{m-2}{\sqrt{m(m-1)}}\left(\sum_{i=1}^{m} b_i^2\right)^{3/2}.$$

Moreover, equality holds in the right-hand (respectively, left-hand) side if and only if $(m-1)$ of the b_i's are nonpositive (respectively, nonnegative) and equal.

By using the above lemmas we obtain the following result on the boundedness of the mean curvature of proper-biharmonic submanifolds with parallel mean curvature vector field in spheres, as well as a partial classification result. We shall see that $|H|$ does not fill out all the interval $(0, 1]$.

Theorem 1.62 ([23]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper-biharmonic immersion. Assume that $m > 2$ and $|H| \in (0, 1)$. Then $|H| \in (0, (m-2)/m]$, and $|H| = (m-2)/m$ if and only if locally $\varphi(M)$ is an open part of a standard product*

$$M_1 \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_1 is a minimal embedded submanifold of $\mathbb{S}^{n-2}(1/\sqrt{2})$. Moreover, if M is complete, then the above decomposition of $\varphi(M)$ holds globally, where M_1 is a complete minimal submanifold of $\mathbb{S}^{n-2}(1/\sqrt{2})$.

Proof. Consider the tensor field Φ associated to M. Since it is traceless, Lemma 1.61 implies that

$$\operatorname{trace}\Phi^3 \geq -\frac{m-2}{\sqrt{m(m-1)}}|\Phi|^3. \tag{1.127}$$

By (1.7), as M is proper-biharmonic with parallel mean curvature vector field, $|A_H|^2 = m|H|^2$ and $\langle A_H, A_\eta \rangle = 0$, for all $\eta \in C(NM)$, η orthogonal to H. From (1.121) we obtain

$$|\Phi|^2 = m|H|^2(1 - |H|^2), \tag{1.128}$$

thus $|\Phi|$ is constant. We can apply Lemma 1.60 and, using (1.127) and (1.128), equation (1.125) leads to

$$0 \geq m^2|H|^3(1-|H|^2)\left(2|H| - \frac{m-2}{\sqrt{m-1}}\sqrt{1-|H|^2}\right),$$

thus $|H| \in (0, \frac{m-2}{m}]$.

The condition $|H| = \frac{m-2}{m}$ holds if and only if $\nabla\Phi = 0$ and we have equality in (1.127). This is equivalent to the fact that $\nabla A_H = 0$ and, by the Okumura Lemma, the principal curvatures in the direction of H are constant functions on M and given by

$$\begin{aligned} \lambda_1 = \ldots = \lambda_{m-1} = \lambda &= \frac{m-2}{m}, \\ \lambda_m = \mu &= -\frac{m-2}{m}. \end{aligned} \tag{1.129}$$

Further, we consider the distributions

$$T_\lambda = \{X \in TM : A_H(X) = \lambda X\}, \qquad \dim T_\lambda = m-1,$$
$$T_\mu = \{X \in TM : A_H(X) = \mu X\}, \qquad \dim T_\mu = 1.$$

One can easily verify that, as A_H is parallel, T_λ and T_μ are mutually orthogonal, smooth, involutive and parallel, and the de Rham decomposition theorem (see [83]) can be applied.

Thus, for every $p_0 \in M$ there exists a neighborhood $U \subset M$ which is isometric to a product $\widetilde{M}_1^{m-1} \times I$, $I = (-\varepsilon, \varepsilon)$, where $\widetilde{M}_1$ is an integral submanifold for T_λ through p_0 and I corresponds to the integral curves of the unit vector field $Y_1 \in T_\mu$ on U. Moreover $\widetilde{M}_1$ is a totally geodesic submanifold in U and the integral curves of Y_1 are geodesics in U. We note that Y_1 is a parallel vector field on U.

In the following, we shall prove that the integral curves of Y_1, thought of as curves in $\mathbb{R}^{n+1}$, are circles of radius $1/\sqrt{2}$, all lying in parallel 2-planes. In order to prove this, consider $\{H/|H|, \eta_a\}_{a=1}^{k-1}$ to be an orthonormal frame field in the normal bundle and $\{X_\alpha\}_{\alpha=1}^{m-1}$ an orthonormal frame field in T_λ, on U. We have

$$\begin{aligned} \operatorname{trace} B(A_H(\cdot), \cdot) &= \sum_{\alpha=1}^{m-1} B(A_H(X_\alpha), X_\alpha) + B(A_H(Y_1), Y_1), \\ &= \lambda m H - 2\lambda B(Y_1, Y_1). \end{aligned}$$

This, together with (1.6) and (1.129), leads to

$$B(Y_1, Y_1) = -\frac{1}{\lambda} H, \tag{1.130}$$

so $|B(Y_1, Y_1)| = 1$. From here, since A_{η_a} and A_H commute, we obtain

$$A_{\eta_a}(Y_1) = 0, \qquad \forall a = 1, \ldots, k-1. \tag{1.131}$$

We also note that

$$\nabla^{\mathbb{S}^n}_{Y_1} B(Y_1, Y_1) = -\frac{1}{\lambda}(\nabla^\perp_{Y_1} H - A_H(Y_1)) = -Y_1. \tag{1.132}$$

Consider $c : I \to U$ to be an integral curve for Y_1 and denote by $\gamma : I \to \mathbb{S}^n$, $\gamma = \imath \circ c$, where $\imath : M \to \mathbb{S}^n$ is the inclusion map. Denote by $E_1 = \dot\gamma = Y_1 \circ \gamma$. Since Y_1 is parallel, c is a geodesic on M and, using equations (1.130) and (1.132), we obtain the following Frenet equations for the curve γ in $\mathbb{S}^n$,

$$\begin{aligned} &\nabla^{\mathbb{S}^n}_{\dot\gamma} E_1 = B(Y_1, Y_1) = -\frac{1}{\lambda} H = E_2, \\ &\nabla^{\mathbb{S}^n}_{\dot\gamma} E_2 = -E_1. \end{aligned} \tag{1.133}$$

Let now $\widetilde\gamma = \jmath \circ \gamma : I \to \mathbb{R}^{n+1}$, where $\jmath : \mathbb{S}^n \to \mathbb{R}^{n+1}$ denotes the inclusion map. Denote by $\widetilde{E}_1 = \dot{\widetilde\gamma} = Y_1 \circ \widetilde\gamma$. From (1.133) we obtain the Frenet equations for $\widetilde\gamma$ in $\mathbb{R}^{n+1}$,

$$\begin{aligned} &\nabla^{\mathbb{R}^{n+1}}_{\dot{\widetilde\gamma}} \widetilde{E}_1 = -\frac{1}{\lambda} H - \widetilde\gamma = \sqrt{2}\widetilde{E}_2, \\ &\nabla^{\mathbb{R}^{n+1}}_{\dot{\widetilde\gamma}} \widetilde{E}_2 = -\sqrt{2}\widetilde{E}_1, \end{aligned}$$

thus $\widetilde{\gamma}$ is a circle of radius $1/\sqrt{2}$ in $\mathbb{R}^{n+1}$ and it lies in a 2-plane with corresponding vector space generated by $\widetilde{E}_1(0)$ and $\widetilde{E}_2(0)$.

Since Y_1 and $-\frac{1}{\lambda}H - \mathrm{x}$, with x the position vector field, are parallel in $\mathbb{R}^{n+1}$ along any curve of $\widetilde{M_1}$, we conclude that the 2-planes determined by the integral curves of Y_1 have the same corresponding vector space, thus are parallel.

Consider the immersions

$$\phi : \widetilde{M_1} \times I \to \mathbb{S}^n,$$

and

$$\widetilde{\phi} = \jmath \circ \phi : \widetilde{M_1} \times I \to \mathbb{R}^{n+1}.$$

Using the fact that $\widetilde{M_1}$ is an integral submanifold of T_λ and (1.131), it is not difficult to verify that $\widetilde{B}(X,Y) = 0$, for all $X \in C(T\widetilde{M_1})$ and $Y \in C(TI)$, thus we can apply the Moore Lemma in [98]. As the 2-planes determined by the integral curves of Y_1 have the same corresponding vector space and by Corollary 1.57, we obtain the orthogonal decomposition

$$\mathbb{R}^{n+1} = \mathbb{R}^{n-1} \oplus \mathbb{R}^2 \tag{1.134}$$

and $U = M_1 \times M_2$, where $M_1^{m-1} \subset \mathbb{R}^{n-1}$ and $M_2 \subset \mathbb{R}^2$ is a circle of radius $1/\sqrt{2}$. We can see that the center of this circle is the origin of $\mathbb{R}^2$. Thus $M_1 \subset \mathbb{S}^{n-2}(1/\sqrt{2}) \subset \mathbb{R}^{n-1}$ and from Theorem 1.6, since U is biharmonic in $\mathbb{S}^n$, we conclude that M_1 is a minimal submanifold in $\mathbb{S}^{n-2}(1/\sqrt{2}) \subset \mathbb{R}^{n-1}$. Consequently, the announced result holds locally.

We can thus conclude that M is an open part of a standard product

$$M_1 \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_1 is a minimal submanifold in $\mathbb{S}^{n-2}(1/\sqrt{2})$.

□

Remark 1.63. The same result of Theorem 1.62 was proved, independently, in [133].

The following consequences for hypersurfaces follow.

Corollary 1.64 ([23]). *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a CMC proper-biharmonic hypersurface with $m > 2$. Then $|H| \in (0,(m-2)/m] \cup \{1\}$. Moreover, $|H| = 1$ if and only if $\varphi(M)$ is an open subset of the small hypersphere $\mathbb{S}^m(1/\sqrt{2})$, and $|H| = (m-2)/m$ if and only if $\varphi(M)$ is an open subset of the standard product $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$.*

Corollary 1.65. *Let $\varphi : M^m \to \mathbb{S}^{m+1}$ be a complete proper-biharmonic hypersurface.*

(i) If $|H| = 1$, then $\varphi(M) = \mathbb{S}^m(1/\sqrt{2})$ and φ is an embedding.

(ii) If $|H| = (m-2)/m$, $m > 2$, then $\varphi(M) = \mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ and the universal cover of M is $\mathbb{S}^{m-1}(1/\sqrt{2}) \times \mathbb{R}$.

If we assume that M is compact and $|B|$ is bounded we obtain the following theorem.

Theorem 1.66 ([16]). *Let $\varphi : M^m \to \mathbb{S}^{m+d}$ be a compact PMC proper-biharmonic immersion with $m \geq 2$, $d \geq 2$ and*

$$m < |B|^2 \leq m\frac{d-1}{2d-3}\left(1 + \frac{3d-4}{d-1}|H|^2 - \frac{m-2}{\sqrt{m-1}}|H|\sqrt{1-|H|^2}\right).$$

(i) If $m = 2$*, then* $|H| = 1$*, and either* $d = 2$*,* $|B|^2 = 6$*,* $\varphi(M^2) = \mathbb{S}^1(1/2) \times \mathbb{S}^1(1/2) \subset \mathbb{S}^3(1/\sqrt{2})$ *or* $d = 3$*,* $|B|^2 = 14/3$*,* $\varphi(M^2)$ *is the Veronese minimal surface in* $\mathbb{S}^3(1/\sqrt{2})$*.*

(ii) If $m > 2$*, then* $|H| = 1$*,* $d = 2$*,* $|B|^2 = 3m$ *and*

$$\varphi(M^m) = \mathbb{S}^{m_1}\left(\sqrt{m_1/(2m)}\right) \times \mathbb{S}^{m_2}\left(\sqrt{m_2/(2m)}\right) \subset \mathbb{S}^{m+1}(1/\sqrt{2}),$$

where $m_1 + m_2 = m$*,* $m_1 \geq 1$ *and* $m_2 \geq 1$*.*

Proof. The result follows from the classification of compact PMC immersions with bounded $|B|^2$ given in Theorem 1.6 of [118]. $\square$

Inspired by the case $|H| = \frac{m-2}{m}$ of Theorem 1.62, in the following we shall study proper-biharmonic submanifolds in $\mathbb{S}^n$ with parallel mean curvature vector field and parallel Weingarten operator associated to the mean curvature vector field.

We shall also need the following general result.

Proposition 1.67 ([23])**.** *Let* $\varphi : M^m \to \mathbb{S}^n$ *be a submanifold with nowhere zero mean curvature vector field. If* $\nabla^\perp H = 0$*,* $\nabla A_H = 0$ *and* A_H *is orthogonal to* A_η*, for all* $\eta \in C(NM)$*,* $\eta \perp H$*, then* M *is either pseudo-umbilical, or it has two distinct principal curvatures in the direction of* H*. Moreover, the principal curvatures in the direction of* H *are solutions of the equation*

$$mt^2 + \left(m - \frac{|A_H|^2}{|H|^2}\right)t - m|H|^2 = 0. \tag{1.135}$$

Proof. As $\nabla A_H = 0$, the principal curvatures in the direction of H are constant on M. Denote by $\{X_i\}_{i=1}^m$ a local orthonormal frame field on M such that $A_H(X_i) = \lambda_i X_i$, $i = 1, \ldots, m$. Clearly, $\sum_{i=1}^m \lambda_i = m|H|^2$.

Since A_H is parallel, $\nabla_X A_H(Y) = A_H(\nabla_X Y)$, thus $R(X,Y)$ and A_H commute for all $X, Y \in C(TM)$. In particular,

$$R(X_i, X_j)A_H(X_j) = A_H(R(X_i, X_j)X_j),$$

and by considering the scalar product with X_j and using the symmetry of A_H, we get

$$(\lambda_i - \lambda_j)\langle R(X_i, X_j)X_j, X_i\rangle = 0, \quad \forall i, j = 1, \ldots, m. \tag{1.136}$$

Consider $\{H/|H|, \eta_a\}_{a=1}^{k-1}$, $k = n - m$, a local orthonormal frame field in the normal bundle of M in $\mathbb{S}^n$. We have

$$B(X_i, X_i) = \frac{\lambda_i}{|H|^2}H + \sum_{a=1}^{k-1}\langle A_{\eta_a}(X_i), X_i\rangle\eta_a, \tag{1.137}$$

and for $\lambda_i \neq \lambda_j$, and so $i \neq j$, as X_i is orthogonal to X_j and $A_H \circ A_{\eta_a} = A_{\eta_a} \circ A_H$, for all $a = 1, \ldots, k-1$, we obtain

$$B(X_i, X_j) = \frac{1}{|H|^2}\langle A_H(X_i), X_j\rangle H + \sum_{a=1}^{k-1}\langle A_{\eta_a}(X_i), X_j\rangle\eta_a = 0. \tag{1.138}$$

By using (1.137) and (1.138) in the Gauss equation for M in $\mathbb{S}^n$, one gets, for $\lambda_i \neq \lambda_j$,

$$\langle R(X_i, X_j)X_j, X_i\rangle = 1 + \frac{\lambda_i\lambda_j}{|H|^2} + \sum_{a=1}^{k-1}\langle A_{\eta_a}(X_i), X_i\rangle\langle A_{\eta_a}(X_j), X_j\rangle. \tag{1.139}$$

In fact, (1.136), together with (1.139), implies

$$(\lambda_i - \lambda_j)\left(1 + \frac{\lambda_i\lambda_j}{|H|^2} + \sum_{a=1}^{k-1}\langle A_{\eta_a}(X_i), X_i\rangle\langle A_{\eta_a}(X_j), X_j\rangle\right) = 0, \tag{1.140}$$

and the above formula holds $\forall i, j = 1, \ldots, m$. Summing on i in (1.140) we obtain

$$\begin{aligned} 0 \quad = \quad & m|H|^2 - \left(m - \frac{|A_H|^2}{|H|^2}\right)\lambda_j - m\lambda_j^2 + \sum_{a=1}^{k-1}\langle A_{\eta_a}, A_H\rangle\langle A_{\eta_a}(X_j), X_j\rangle \\ & - \sum_{a=1}^{k-1}\operatorname{trace} A_{\eta_a}\langle A_{\eta_a}(X_j), A_H(X_j)\rangle. \end{aligned}$$

Since $\operatorname{trace} A_{\eta_a} = 0$ and $\langle A_H, A_{\eta_a}\rangle = 0$, for all $a = 1, \ldots, k-1$, we conclude the proof. $\square$

Corollary 1.68 ([23])**.** *Let M^m, $m > 2$, be a proper-biharmonic submanifold in $\mathbb{S}^n$. If $\nabla^\perp H = 0$, $\nabla A_H = 0$ and $|H| \in (0, \frac{m-2}{m}]$, then M has two distinct principal curvatures λ and μ in the direction of H, of different multiplicities m_1 and m_2, respectively, and*

$$\begin{cases} \lambda = \dfrac{m_1 - m_2}{m} \\ \mu = -\dfrac{m_1 - m_2}{m} \\ |H| = \dfrac{|m_1 - m_2|}{m}. \end{cases} \tag{1.141}$$

Proof. Since M is proper-biharmonic, all the hypotheses of Proposition 1.67 are satisfied. Taking into account (1.7), from (1.135) follows that the principal curvatures of M in the direction of H satisfy the equation $t^2 = |H|^2$. As $|H| \in (0, \frac{m-2}{m}]$, M cannot be pseudo-umbilical, thus it has two distinct principal curvatures $\lambda = -\mu \neq 0$ in the direction of H. If m_1 denotes the multiplicity of λ and m_2 the multiplicity of μ, from $\operatorname{trace} A_H = m|H|^2$ we have $(m_1 - m_2)\lambda = m\lambda^2$. Since $\lambda \neq 0$, we obtain (1.141). Notice also that $m_1 \neq m_2$. $\square$

The case $|H| = \dfrac{m-2}{m}$ was solved in Theorem 1.62, thus we shall consider now only the case $|H| \in (0, \frac{m-2}{m})$. Since $|H| = \dfrac{|m_1 - m_2|}{m}$, $m_1 \neq m_2$ we obtain the following.

Corollary 1.69 ([23])**.** *Let $\varphi : M^m \to \mathbb{S}^n$, $m \in \{3, 4\}$, be a PMC proper-biharmonic immersion with $\nabla A_H = 0$. Then $|H| \in \{(m-2)/m, 1\}$. Moreover, if $|H| = (m-2)/m$, then locally $\varphi(M)$ is an open part of a standard product*

$$M_1 \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_1 is a minimal embedded submanifold of $\mathbb{S}^{n-2}(1/\sqrt{2})$, and if $|H|=1$, then φ induces a minimal immersion of M into $\mathbb{S}^{n-1}(1/\sqrt{2})$.

We are left with the case $m \geq 5$ and $m_1 \geq 2$, $m_2 \geq 2$ and we are able to prove the following result.

Theorem 1.70 ([23]). *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper-biharmonic immersion with $\nabla A_H = 0$. Assume that $|H| \in (0,(m-2)/m)$. Then, $m > 4$ and, locally,*

$$\varphi(M) = M_1^{m_1} \times M_2^{m_2} \subset \mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_i is a minimal embedded submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $m_i \geq 2$, $i=1,2$, $m_1+m_2 = m$, $m_1 \neq m_2$, $n_1+n_2 = n-1$. In this case $|H| = |m_1-m_2|/m$. Moreover, if M is complete, then the above decomposition of $\varphi(M)$ holds globally, where M_i is a complete minimal submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $i=1,2$.

Proof. We are in the hypotheses of Corollary 1.68, thus A_H has two distinct eigenvalues $\lambda = \frac{m_1-m_2}{m}$ and $\mu = -\frac{m_1-m_2}{m}$. Consider the distributions

$$\begin{aligned} T_\lambda &= \{X \in TM : A_H(X) = \lambda X\}, \qquad \dim T_\lambda = m_1, \\ T_\mu &= \{X \in TM : A_H(X) = \mu X\}, \qquad \dim T_\mu = m_2. \end{aligned}$$

As A_H is parallel, T_λ and T_μ are mutually orthogonal, smooth, involutive and parallel, and from the de Rham decomposition theorem follows that for every $p_0 \in M$ there exists a neighborhood $U \subset M$ which is isometric to a product $\widetilde{M}_1^{m_1} \times \widetilde{M}_2^{m_2}$, such that the submanifolds which are parallel to $\widetilde{M}_1$ in $\widetilde{M}_1 \times \widetilde{M}_2$ correspond to integral submanifolds for T_λ and the submanifolds which are parallel to $\widetilde{M}_2$ correspond to integral submanifolds for T_μ.

Consider the immersions

$$\phi : \widetilde{M}_1 \times \widetilde{M}_2 \to \mathbb{S}^n,$$

and

$$\widetilde{\phi} = \jmath \circ \phi : \widetilde{M}_1 \times \widetilde{M}_2 \to \mathbb{R}^{n+1}.$$

It can be easily verified that $\widetilde{B}(X,Y) = B(X,Y)$, for all $X \in C(T\widetilde{M}_1)$ and $Y \in C(T\widetilde{M}_2)$. Since $A_H \circ A_\eta = A_\eta \circ A_H$ for all $\eta \in C(NM)$, we have that T_λ and T_μ are invariant subspaces for A_η, for all $\eta \in C(NM)$, thus

$$\langle B(X,Y), \eta \rangle = \langle A_\eta(X), Y \rangle = 0, \quad \forall \eta \in C(NM).$$

Thus $\widetilde{B}(X,Y) = 0$, for all $X \in C(T\widetilde{M}_1)$ and $Y \in C(T\widetilde{M}_2)$, and we can apply the Moore Lemma. In this way we have an orthogonal decomposition $\mathbb{R}^{n+1} = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1+1} \oplus \mathbb{R}^{n_2+1}$ and $\widetilde{\phi}$ is a product immersion. From Corollary 1.57, since $|H| \neq 1$, follows that $n_0 = 0$. Thus

$$\widetilde{\phi} = \widetilde{\phi}_1 \times \widetilde{\phi}_2 : \widetilde{M}_1 \times \widetilde{M}_2 \to \mathbb{R}^{n_1+1} \oplus \mathbb{R}^{n_2+1}.$$

We denote by $M_1 = \widetilde{\phi}_1(\widetilde{M}_1) \subset \mathbb{R}^{n_1+1}$, $M_2 = \widetilde{\phi}_2(\widetilde{M}_2) \subset \mathbb{R}^{n_2+1}$ and we have $U = M_1 \times M_2 \subset \mathbb{S}^n$.

Consider now $\{X_\alpha\}_{\alpha=1}^{m_1}$ an orthonormal frame field in T_λ and $\{Y_\ell\}_{\ell=1}^{m_2}$ an orthonormal frame field in T_μ, on U. From (1.6), by using the fact that $\lambda = -\mu = \dfrac{m_1 - m_2}{m}$, we obtain

$$\sum_{\alpha=1}^{m_1} B(X_\alpha, X_\alpha) = \frac{m_1}{\lambda} H, \qquad \sum_{\ell=1}^{m_2} B(Y_\ell, Y_\ell) = -\frac{m_2}{\lambda} H. \tag{1.142}$$

Since $\nabla^\perp H = 0$, from (1.142) follows that $M_1 \times \{p_2\}$ is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{R}^{n+1}$, for any $p_2 \in M_2$. But $M_1 \times \{p_2\}$ is included in $\mathbb{R}^{n_1+1} \times \{p_2\}$ which is totally geodesic in $\mathbb{R}^{n+1}$, thus M_1 is pseudo-umbilical and with parallel mean curvature vector field in $\mathbb{R}^{n_1+1}$. This implies that M_1 is minimal in $\mathbb{R}^{n_1+1}$ or minimal in a hypersphere of $\mathbb{R}^{n_1+1}$. The first case leads to a contradiction, since $M_1 \times \{p_2\} \subset \mathbb{S}^n$ and cannot be minimal in $\mathbb{R}^{n+1}$. Thus M_1 is minimal in a hypersphere $\mathbb{S}^{n_1}_{c_1}(r_1) \subset \mathbb{R}^{n_1+1}$, where $c_1 \in \mathbb{R}^{n_1+1}$ denotes the center of the hypersphere.

In the following we will show that $c_1 = 0$. Since $U \subset \mathbb{S}^n$ and $M_1 \subset \mathbb{S}^{n_1}_{c_1}(r_1)$, we get $|p_1|^2 + |p_2|^2 = 1$ and $|p_1 - c_1|^2 = r_1^2$, for all $p_1 \in M_1$. This implies $\langle p_1, c_1 \rangle = $ constant for all $p_1 \in M_1$. Thus $\langle u_1, c_1 \rangle = 0$, for all $u_1 \in T_{p_1} M_1$ and for all $p_1 \in M_1$. From the Moore Lemma follows that $c_1 = 0$, thus $M_1 \subset \mathbb{S}^{n_1}(r_1) \subset \mathbb{R}^{n_1+1}$.

From (1.142) also follows that the mean curvature of $M_1 \times \{p_2\}$ in $\mathbb{S}^n$ is 1, so its mean curvature in $\mathbb{R}^{n+1}$ is $\sqrt{2}$. As $\mathbb{R}^{n_1+1} \times \{p_2\}$ is totally geodesic in $\mathbb{R}^{n+1}$ it follows that the mean curvature of M_1 in $\mathbb{R}^{n_1+1}$ is $\sqrt{2}$ too. Further, as M_1 is minimal in $\mathbb{S}^{n_1}(r_1)$, we get $r_1 = 1/\sqrt{2}$.

Analogously, M_2 is minimal in a hypersphere $\mathbb{S}^{n_2}(1/\sqrt{2})$ in $\mathbb{R}^{n_2+1}$, and we conclude the proof.

□

Remark 1.71. In the case of a non-minimal hypersurface the hypotheses $\nabla^\perp H = 0$ and $\nabla A_H = 0$ are equivalent to $\nabla^\perp B = 0$, i.e. the hypersurface is parallel. Such hypersurfaces have at most two principal curvatures and the proper-biharmonic hypersurfaces with at most two principal curvatures in $\mathbb{S}^n$ are those given by (1.8) and (1.9) (see [21]).

If one searches for a relaxation of the hypothesis $\nabla A_H = 0$ in Theorem 1.70, natural candidates would be $RA_H = 0$ (see, for example, [115]), or M has at most two distinct principal curvatures in the direction of H everywhere. But the following can be proved.

Proposition 1.72 ([23])**.** *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper-biharmonic immersion. The following statements are equivalent:*

(i) $RA_H = 0$, *where* $(RA_H)(X, Y, Z) = (R(X, Y)A_H)(Z)$,

(ii) *M has at most two distinct principal curvatures in the direction of H everywhere,*

(iii) $\nabla A_H = 0$.

Another restriction which leads to $\nabla A_H = 0$, thus to a classification result for PMC proper-biharmonic immersions in spheres, concerns the sectional curvature.

Proposition 1.73. *Let $\varphi : M^m \to \mathbb{S}^n$ be a PMC proper-biharmonic immersion with non-negative sectional curvature. Then $\nabla A_H = 0$. Moreover, if there exists $p \in M$ such that* $\mathrm{Riem}^M(p) > 0$, *then φ is pseudo-umbilical.*

Proof. Since M is PMC, $|H|=$ constant and A_H is a Codazzi operator. Moreover, as M is also biharmonic, $|A_H|=$ constant and equation (2.8) in [49] applied for A_H, implies

$$0=|\nabla A_H|^2+\sum_{i<j} R_{ijij}(\lambda_i-\lambda_j)^2,$$

where $\{\lambda_i\}_{i=1}^n$ denote the principal curvatures of M in the direction of H. The conclusion is now immediate. □

We should note that there exist examples of proper-biharmonic submanifolds of $\mathbb{S}^5$ and $\mathbb{S}^7$ which are not PMC but with $\nabla A_H=0$ (see [120] and [67]).

1.6 Parallel biharmonic immersions in $\mathbb{S}^n$

An immersed submanifold is said to be *parallel* if its second fundamental form B is parallel, that is $\nabla^{\perp}B=0$.

In the following we give the classification for proper-biharmonic parallel immersed surfaces in $\mathbb{S}^n$.

Theorem 1.74 ([16])**.** *Let $\varphi: M^2\to\mathbb{S}^n$ be a parallel surface in $\mathbb{S}^n$. If φ is proper-biharmonic, then the codimension can be reduced to* 3 *and $\varphi(M)$ is an open part of either*

(i) *a totally umbilical sphere $\mathbb{S}^2(1/\sqrt{2})$ lying in a totally geodesic $\mathbb{S}^3\subset\mathbb{S}^5$, or*

(ii) *the minimal flat torus $\mathbb{S}^1(1/2)\times\mathbb{S}^1(1/2)\subset\mathbb{S}^3(1/\sqrt{2})$; $\varphi(M)$ lies in a totally geodesic $\mathbb{S}^4\subset\mathbb{S}^5$, or*

(iii) *the minimal Veronese surface in $\mathbb{S}^4(1/\sqrt{2})\subset\mathbb{S}^5$.*

Proof. The proof relies on the fact that parallel submanifolds in $\mathbb{S}^n$ are classified in the following three categories (see, for example, [40]):

(a) a totally umbilical sphere $\mathbb{S}^2(r)$ lying in a totally geodesic $\mathbb{S}^3\subset\mathbb{S}^n$;

(b) a flat torus lying in a totally geodesic $\mathbb{S}^4\subset\mathbb{S}^n$ defined by

$$(0,\dots,0,a\cos u,a\sin u,b\cos v,b\sin v,\sqrt{1-a^2-b^2}),\quad a^2+b^2\le 1;$$

(c) a surface of positive constant curvature lying in a totally geodesic $\mathbb{S}^5\subset\mathbb{S}^n$ defined by

$$r\left(0,\dots,0,\frac{vw}{\sqrt{3}},\frac{uw}{\sqrt{3}},\frac{uv}{\sqrt{3}},\frac{u^2-v^2}{2\sqrt{3}},\frac{u^2+v^2-2w^2}{6},\frac{\sqrt{1-r^2}}{r}\right),$$

with $u^2+v^2+w^2=3$ and $0<r\le 1$.

In case (a) the biharmonicity implies directly (i). Requiring the immersion in (b) to be biharmonic and using [21, Corollary 5.5] we get that $\sqrt{a^2+b^2}=1/2$ and then (ii) follows. The immersion in (c) induces a minimal immersion of the surface in the hypersphere $\mathbb{S}^4(r)\subset\mathbb{S}^5$. Then, applying [30, Theorem 3.5], the immersion in (c) reduces to that in (iii). □

In all three cases of Theorem 1.74, φ is of type **B3** and thus its mean curvature is 1. In the higher dimensional case we know, from Theorem 1.7, that if $|H| = 1$, then φ is of type **B3**. Moreover, if we assume that φ is also parallel, then the induced minimal immersion in $\mathbb{S}^{n-1}(1/\sqrt{2})$ is parallel as well.

If $\nabla^{\perp} B = 0$, then $\nabla^{\perp} H = 0$ and $\nabla A_H = 0$. Therefore Theorem 1.62 and Theorem 1.70 hold also for parallel proper-biharmonic immersions in $\mathbb{S}^n$. From this and Theorem 1.74, in order to classify all parallel proper-biharmonic immersions in $\mathbb{S}^n$, we are left with the case when $m > 2$ and $|H| \in (0,1)$.

Theorem 1.75 ([16])**.** *Let $\varphi : M^m \to \mathbb{S}^n$ be a parallel proper-biharmonic immersion. Assume that $m > 2$ and $|H| \in (0,1)$. Then $|H| \in (0,(m-2)/m]$. Moreover:*

(i) $|H| = (m-2)/m$ if and only if locally $\varphi(M)$ is an open part of a standard product

$$M_1 \times \mathbb{S}^1(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_1 is a parallel minimal embedded submanifold of $\mathbb{S}^{n-2}(1/\sqrt{2})$;

(ii) $|H| \in (0,(m-2)/m)$ if and only if $m > 4$ and, locally,

$$\varphi(M) = M_1^{m_1} \times M_2^{m_2} \subset \mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2}) \subset \mathbb{S}^n,$$

where M_i is a parallel minimal embedded submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $m_i \geq 2$, $i = 1,2$, $m_1 + m_2 = m$, $m_1 \neq m_2$, $n_1 + n_2 = n - 1$.

Proof. We only have to prove that M_i is a parallel minimal submanifold of $\mathbb{S}^{n_i}(1/\sqrt{2})$, $m_i \geq 2$. For this, denote by B^i the second fundamental form of M_i in $\mathbb{S}^{n_i}(1/\sqrt{2})$, $i = 1,2$. If B denotes the second fundamental form of $M_1 \times M_2$ in $\mathbb{S}^n$, it is easy to verify, using the expression of the second fundamental form of $\mathbb{S}^{n_1}(1/\sqrt{2}) \times \mathbb{S}^{n_2}(1/\sqrt{2})$ in $\mathbb{S}^n$, that

$$(\nabla^{\perp}_{(X_1,X_2)} B)((Y_1,Y_2),(Z_1,Z_2)) = ((\nabla^{\perp}_{X_1} B^1)(Y_1,Z_1), (\nabla^{\perp}_{X_2} B^2)(Y_2,Z_2)),$$

for all $X_1, Y_1, Z_1 \in C(TM_1)$, $X_2, Y_2, Z_2 \in C(TM_2)$. Consequently, $M_1 \times M_2$ is parallel in $\mathbb{S}^n$ if and only if M_i is parallel in $\mathbb{S}^{n_i}(1/\sqrt{2})$, $i = 1,2$. □

1.7 Open problems

We list some open problems and conjectures that seem to be natural.

Conjecture 1. *The only proper-biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are the open parts of hyperspheres $\mathbb{S}^m(1/\sqrt{2})$ or of the standard products of spheres $\mathbb{S}^{m_1}(1/\sqrt{2}) \times \mathbb{S}^{m_2}(1/\sqrt{2})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Taking into account the results presented in this chapter, we have a series of statements equivalent to Conjecture 1:

1. A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ has at most two principal curvatures everywhere.

2. A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is parallel.

3. A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is CMC and has non-negative sectional curvature.

4. A proper-biharmonic hypersurface in $\mathbb{S}^{m+1}$ is isoparametric.

One can also state the following intermediate conjecture.

Conjecture 2. *The proper-biharmonic hypersurfaces in* $\mathbb{S}^{m+1}$ *are CMC.*

Related to PMC immersions and, in particular, to Theorem 1.70, we propose the following problem.

Problem 1. *Find a PMC proper-biharmonic immersion* $\varphi : M^m \to \mathbb{S}^n$ *such that* A_H *is not parallel.*

Chapter 2

Biharmonic submanifolds in complex space forms

2.1 Introduction

In the first part of Chapter 2 we obtain some general properties of proper-biharmonic submanifolds with constant mean curvature, or parallel mean curvature vector field, of the complex projective space endowed with the standard Fubini-Study metric. When the ambient space is a complex space form of nonpositive holomorphic curvature we obtain non-existence results.

In the second part we consider the Hopf map defined as the restriction of the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ to the sphere $\mathbb{S}^{2n+1}$, which defines a Riemannian submersion. For a real submanifold $\bar{M}$ of $\mathbb{C}P^n$ we denote by $M := \pi^{-1}(\bar{M})$ the Hopf-tube over $\bar{M}$. We obtain the formula which relates the bitension field of the inclusion of $\bar{M}$ in $\mathbb{C}P^n$ and the bitension field of the inclusion of $M = \pi^{-1}(\bar{M})$ in $\mathbb{S}^{2n+1}$ (Theorem 2.12). Using this formula we are able to produce a new class of proper-biharmonic submanifolds $\bar{M}$ of $\mathbb{C}P^n$ when M is of "Clifford type" (Theorem 2.18), and to reobtain a result in [139] when M is a product of circles (Theorem 2.26).

We note that $\bar{M}$ is minimal (harmonic) in $\mathbb{C}P^n$ if and only if M is minimal in $\mathbb{S}^{2n+1}$ (see [86]) but, for what concerns the biharmonicity, the result does not hold anymore.

In the last part of the chapter we focus on the geometry of proper-biharmonic curves of $\mathbb{C}P^n$. We characterize all proper-biharmonic curves of $\mathbb{C}P^n$ in terms of their curvatures and complex torsions. Then, using the classification of holomorphic helices of $\mathbb{C}P^2$ given in [91], we determine all proper-biharmonic curves of $\mathbb{C}P^2$ (Theorem 2.41).

2.2 Biharmonic submanifolds of complex space forms

Let $\mathbb{E}^n_{\mathbb{C}}(4c)$ be a complex space form of holomorphic sectional curvature $4c$. Let us denote by $\bar{J}$ the complex structure and by $\langle , \rangle$ the Riemannian metric on $\mathbb{E}^n_{\mathbb{C}}(4c)$. Then its curvature operator is given, for vector fields X, Y and Z, by

$$\begin{aligned} R^{\mathbb{E}^n_{\mathbb{C}}(4c)}(X,Y)Z \quad = \quad & c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y \\ & + \langle \bar{J}Y,Z\rangle \bar{J}X - \langle \bar{J}X,Z\rangle \bar{J}Y + 2\langle X,\bar{J}Y\rangle \bar{J}Z\}. \end{aligned} \tag{2.1}$$

Let now

$$\bar{\jmath} : \bar{M}^{\bar{m}} \to \mathbb{E}^n_{\mathbb{C}}(4c)$$

be the canonical inclusion of a submanifold $\bar{M}$ in $\mathbb{E}^n_{\mathbb{C}}(4c)$ of real dimension $\bar{m}$. Then the bitension field becomes

$$\tau_2(\bar{\jmath}) = -\bar{m}\{\Delta^{\bar{\jmath}}\bar{H} - c\bar{m}\bar{H} + 3c\bar{J}\left(\bar{J}\bar{H}\right)^{\top}\}, \tag{2.2}$$

where $\bar{H}$ denotes the mean curvature vector field, $\Delta^{\bar{\jmath}}$ is the rough Laplacian, and $()^{\top}$ denotes the tangential component to $\bar{M}$. The overbar notation will be justified in the next section. If we assume that $\bar{J}\bar{H}$ is tangent to $\bar{M}$, then (2.2) simplifies to

$$\tau_2(\bar{\jmath}) = -\bar{m}\{\Delta^{\bar{\jmath}}\bar{H} - c(\bar{m}+3)\bar{H}\}. \tag{2.3}$$

Decomposing (2.3) with respect to its tangential and normal components we get

Proposition 2.1 ([66]). *Let $\bar{M}$ be a real submanifold of $\mathbb{E}^n_{\mathbb{C}}(4c)$ of dimension $\bar{m}$ such that $\bar{J}\bar{H}$ is tangent to $\bar{M}$. Then $\bar{M}$ is biharmonic if and only if*

$$\begin{cases} \Delta^{\perp}\bar{H} + \operatorname{trace}\bar{B}(\cdot, \bar{A}_{\bar{H}}(\cdot)) - c(\bar{m}+3)\bar{H} = 0 \\ 4\operatorname{trace}\bar{A}_{\nabla^{\perp}_{(\cdot)}\bar{H}}(\cdot) + \bar{m}\operatorname{grad}(|\bar{H}|^2) = 0 \end{cases}, \tag{2.4}$$

where $\bar{A}$ denotes the Weingarten operator, $\bar{B}$ the second fundamental form, $\bar{H}$ the mean curvature vector field, $\nabla^{\perp}$ and $\Delta^{\perp}$ the connection and the Laplacian in the normal bundle of $\bar{M}$ in $\mathbb{E}^n_{\mathbb{C}}(4c)$.

If $\bar{M}$ is a hypersurface, then $\bar{J}\bar{H}$ is tangent to $\bar{M}$, and the previous proposition gives the following result of [77, 78].

Corollary 2.2. *Let $\bar{M}$ be a real hypersurface of $\mathbb{E}^n_{\mathbb{C}}(4c)$ of non-zero constant mean curvature. Then it is proper-biharmonic if and only if*

$$|\bar{B}|^2 = 2c(n+1).$$

Proposition 2.1 can be applied also in the case of Lagrangian submanifolds. We recall here that $\bar{M}$ is called a Lagrangian submanifold if $\dim\bar{M} = n$ and $\bar{\jmath}^*\Omega = 0$, where Ω is the fundamental 2-form on $\mathbb{E}^n_{\mathbb{C}}(4c)$ defined by $\Omega(X,Y) = \langle X, \bar{J}Y\rangle$, for any vector fields X and Y tangent to $\mathbb{E}^n_{\mathbb{C}}(4c)$.

Corollary 2.3 ([66]). *Let $\bar{M}$ be a Lagrangian submanifold of $\mathbb{E}^n_{\mathbb{C}}(4c)$ with parallel mean curvature vector field. Then it is biharmonic if and only if*

$$\operatorname{trace}\bar{B}(\cdot, \bar{A}_{\bar{H}}(\cdot)) = c(n+3)\bar{H}.$$

In the sequel we shall consider only the case of complex space forms with positive holomorphic sectional curvature. A partial motivation of this fact is that Corollary 2.2 rules out the case $c \leq 0$. As usual, we consider the complex projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, endowed with the Fubini-Study metric, as a model for the complex space form of positive constant holomorphic sectional curvature 4.

Proposition 2.4 ([66]). *Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ such that $J\bar{H}$ is tangent to $\bar{M}$. Assume that it has non-zero constant mean curvature. We have*

(a) If $\bar{M}$ is proper-biharmonic, then $|\bar{H}|^2 \in (0, \frac{\bar{m}+3}{\bar{m}}]$.

(b) If $|\bar{H}|^2 = \frac{\bar{m}+3}{\bar{m}}$, then $\bar{M}$ is proper-biharmonic if and only if it is pseudo-umbilical and $\nabla^\perp \bar{H} = 0$.

Proof. Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ such that $J\bar{H}$ is tangent to $\bar{M}$. Assume that it has non-zero constant mean curvature, and it is biharmonic. As $\bar{M}$ is biharmonic we have

$$\Delta^\perp \bar{H} = (\bar{m}+3)\bar{H} - \operatorname{trace} \bar{B}(\cdot, \bar{A}_{\bar{H}}(\cdot)),$$

so

$$\langle \Delta^\perp \bar{H}, \bar{H} \rangle = (\bar{m}+3)|\bar{H}|^2 - \sum_{i=1}^{\bar{m}} \langle \bar{B}(X_i, \bar{A}_{\bar{H}}(X_i)), \bar{H} \rangle = (\bar{m}+3)|\bar{H}|^2 - |\bar{A}_{\bar{H}}|^2.$$

Replacing in the Weitzenböck formula (see, for example, [59])

$$\frac{1}{2}\Delta |\bar{H}|^2 = \langle \Delta^\perp \bar{H}, \bar{H} \rangle - |\nabla^\perp \bar{H}|^2$$

the expression of $\langle \Delta^\perp \bar{H}, \bar{H} \rangle$, and using the fact that $|\bar{H}|$ is constant, we obtain

$$(\bar{m}+3)|\bar{H}|^2 = |\bar{A}_{\bar{H}}|^2 + |\nabla^\perp \bar{H}|^2. \tag{2.5}$$

Let p be an arbitrary point of $\bar{M}$ and let $\{X_i\}_{i=1}^{\bar{m}}$ be an orthonormal basis of $T_p\bar{M}$ such that $\bar{A}_{\bar{H}}(X_i) = \lambda_i X_i$. We have

$$\lambda_i = \langle \bar{A}_{\bar{H}}(X_i), X_i \rangle = \langle \bar{B}(X_i, X_i), \bar{H} \rangle$$

which implies

$$\sum_{i=1}^{\bar{m}} \lambda_i = \bar{m}|\bar{H}|^2$$

or, equivalently,

$$|\bar{H}|^2 = \frac{\sum_{i=1}^{\bar{m}} \lambda_i}{\bar{m}}.$$

Then the square of the norm of $\bar{A}_{\bar{H}}$ becomes

$$|\bar{A}_{\bar{H}}|^2 = \sum_{i=1}^{\bar{m}} \langle \bar{A}_{\bar{H}}(X_i), \bar{A}_{\bar{H}}(X_i) \rangle = \sum_{i=1}^{\bar{m}} (\lambda_i)^2.$$

Replacing in (2.5) we get

$$\frac{\bar{m}+3}{\bar{m}} \sum_{i=1}^{\bar{m}} \lambda_i = \sum_{i=1}^{\bar{m}} (\lambda_i)^2 + |\nabla^\perp \bar{H}|^2 \geq \frac{(\sum_{i=1}^{\bar{m}} \lambda_i)^2}{\bar{m}} + |\nabla^\perp \bar{H}|^2.$$

Therefore

$$(\bar{m}+3)|\bar{H}|^2 \geq \bar{m}|\bar{H}|^4 + |\nabla^\perp \bar{H}|^2 \geq \bar{m}|\bar{H}|^4,$$

so

$$|\bar{H}|^2 \in (0, \frac{\bar{m}+3}{\bar{m}}].$$

(b) If $|\bar{H}|^2 = \frac{\bar{m}+3}{\bar{m}}$ and $\bar{M}$ is biharmonic, the above inequalities become equalities, and therefore $\lambda_1 = \cdots = \lambda_{\bar{m}}$ and $\nabla^\perp \bar{H} = 0$, i.e. $\bar{M}$ is pseudo-umbilical and $\nabla^\perp \bar{H} = 0$. Conversely, it is clear that if $|\bar{H}|^2 = \frac{\bar{m}+3}{\bar{m}}$ and $\bar{M}$ is pseudo-umbilical with $\nabla^\perp \bar{H} = 0$, then $\bar{M}$ is proper-biharmonic. □

Remark 2.5. We shall see in Proposition 2.28 that the upper bound of $|\bar{H}|^2$ is reached in the case of curves.

Proposition 2.6 ([66])**.** *Let $\bar{M}$ be a proper-biharmonic real hypersurface of $\mathbb{C}P^n$ of constant mean curvature $|\bar{H}|$. Then its scalar curvature $s^{\bar{M}}$ is constant and given by*

$$s^{\bar{M}} = 4n^2 - 2n - 4 + (2n-1)^2|\bar{H}|^2.$$

Proof. Let $\bar{M}^{2n-1}$ be a proper-biharmonic real hypersurface of $\mathbb{C}P^n$ with constant mean curvature, so $|\bar{B}|^2 = 2(n+1)$.

The Gauss equation for the submanifold $\bar{M}$ of $\mathbb{C}P^n$ is

$$\begin{aligned} \langle R^{\bar{M}}(X,Y)Z,T\rangle &= \langle R^{\mathbb{C}P^n}(X,Y)Z,T\rangle \\ &\quad -\langle \bar{B}(Y,T),\bar{B}(X,Z)\rangle + \langle \bar{B}(X,T),\bar{B}(Y,Z)\rangle, \end{aligned} \tag{2.6}$$

where $R^{\bar{M}}$ is the curvature tensor field of $\bar{M}$.

Let us denote by $\mathrm{Ricci}^{\bar{M}}(X,Y) = \mathrm{trace}\{Z \to R^{\bar{M}}(Z,X)Y\}$ the Ricci tensor. Computing (2.6) for $X = T = X_i$, where $\{X_i\}_{i=1}^{2n-1}$ is a local orthonormal frame field, we have

$$\begin{aligned} \langle R^{\bar{M}}(X_i,Y)Z,X_i\rangle &= \langle\langle Z,Y\rangle X_i - \langle Z,X_i\rangle Y, X_i\rangle \\ &\quad +\langle\langle \bar{J}Y,Z\rangle \bar{J}X_i, X_i\rangle - \langle\langle \bar{J}X_i,Z\rangle \bar{J}Y,X_i\rangle \\ &\quad +2\langle\langle X_i,\bar{J}Y\rangle \bar{J}Z,X_i\rangle \\ &\quad -\langle \bar{B}(Y,X_i),\bar{B}(X_i,Z)\rangle + \langle \bar{B}(X_i,X_i),\bar{B}(Y,Z)\rangle \\ &= \langle Z,Y\rangle - \langle Z,X_i\rangle\langle Y,X_i\rangle \\ &\quad +\langle \bar{J}Y,Z\rangle\langle \bar{J}X_i,X_i\rangle - \langle \bar{J}X_i,Z\rangle\langle \bar{J}Y,X_i\rangle \\ &\quad +2\langle X_i,\bar{J}Y\rangle\langle \bar{J}Z,X_i\rangle - \langle \bar{B}(Y,X_i),\bar{B}(Z,X_i)\rangle \\ &\quad +\langle \bar{B}(X_i,X_i),\bar{B}(Y,Z)\rangle \\ &= \langle Z,Y\rangle - \langle Z,X_i\rangle\langle Y,X_i\rangle + 3\langle \bar{J}Z,X_i\rangle\langle \bar{J}Y,X_i\rangle \\ &\quad -\langle \bar{A}(Y),X_i\rangle\langle \bar{A}(Z),X_i\rangle + \langle \bar{B}(X_i,X_i),\bar{B}(Y,Z)\rangle, \end{aligned}$$

where $\bar{H} = |\bar{H}|\bar{\eta}$ and $\bar{A} = \bar{A}_{\bar{\eta}}$. Therefore

$$\begin{aligned} \mathrm{Ricci}^{\bar{M}}(Y,Z) &= \sum_{i=1}^{2n-1} \langle R^{\bar{M}}(X_i,Y)Z,X_i\rangle \\ &= (2n-1)\langle Z,Y\rangle - \langle Z,Y\rangle + 3\langle (\bar{J}Z)^\top,(\bar{J}Y)^\top\rangle \\ &\quad -\langle \bar{A}(Y),\bar{A}(Z)\rangle + (2n-1)|\bar{H}|\langle \bar{A}(Y),Z\rangle. \end{aligned}$$

Now,

$$\begin{aligned}\langle \bar{J}Z, \bar{J}Y\rangle &= \langle Z, Y\rangle \\ &= \langle (\bar{J}Z)^\top + \langle \bar{J}Z, \bar{\eta}\rangle\bar{\eta}, (\bar{J}Y)^\top + \langle \bar{J}Y, \bar{\eta}\rangle\bar{\eta}\rangle \\ &= \langle (\bar{J}Z)^\top, (\bar{J}Y)^\top\rangle + \langle \bar{J}Z, \bar{\eta}\rangle\langle \bar{J}Y, \bar{\eta}\rangle,\end{aligned}$$

which implies

$$\langle (\bar{J}Z)^\top, (\bar{J}Y)^\top\rangle = \langle Z, Y\rangle - \langle Z, \bar{J}\bar{\eta}\rangle\langle Y, \bar{J}\bar{\eta}\rangle.$$

Replacing in the above expression of the Ricci tensor, we get

$$\begin{aligned}\mathrm{Ricci}^{\bar{M}}(Y,Z) &= 2(n-1)\langle Z, Y\rangle + 3\{\langle Y, Z\rangle - \langle Z, \bar{J}\bar{\eta}\rangle\langle Y, \bar{J}\bar{\eta}\rangle\} \\ &\quad -\langle \bar{A}(Y), \bar{A}(Z)\rangle + (2n-1)|\bar{H}|\langle \bar{A}(Y), Z\rangle.\end{aligned}$$

Finally, taking the trace, we have

$$\begin{aligned}s^{\bar{M}} &= \sum_{i=1}^{2n-1} \rho^{\bar{M}}(X_i, X_i) = 2(n-1)(2n-1) + 3(2n-1) \\ &\quad -|\bar{J}\bar{\eta}|^2 - |\bar{A}|^2 + (2n-1)^2|\bar{H}|^2 \\ &= (2n-2+3)(2n-1) - 1 - 2(n+1) + (2n-1)^2|\bar{H}|^2 \\ &= 4n^2 - 2n - 4 + (2n-1)^2|\bar{H}|^2.\end{aligned}$$

□

Another important family of submanifolds of $\mathbb{C}P^n$ is that consisting of the submanifolds for which $\bar{J}\bar{H}$ is normal to $\bar{M}$. In this case, using an argument similar to the case when $\bar{J}\bar{H}$ is tangent to $\bar{M}$, we have the following result.

Proposition 2.7 ([66]). *Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ such that $\bar{J}\bar{H}$ is normal to $\bar{M}$. Then $\bar{M}$ is biharmonic if and only if*

$$\begin{cases}\Delta^\perp \bar{H} + \operatorname{trace} \bar{B}(\cdot, \bar{A}_{\bar{H}}(\cdot)) - \bar{m}\bar{H} = 0 \\ 4\operatorname{trace} \bar{A}_{\nabla^\perp_{(\cdot)}\bar{H}}(\cdot) + \bar{m}\operatorname{grad}(|\bar{H}|^2) = 0\end{cases}. \tag{2.7}$$

Moreover, if $\bar{J}\bar{H}$ is normal to $\bar{M}$ and $\bar{M}$ has parallel mean curvature, then $\bar{M}$ is biharmonic if and only if

$$\operatorname{trace} \bar{B}(\cdot, \bar{A}_{\bar{H}}(\cdot)) = \bar{m}\bar{H}.$$

Also in this case, if the mean curvature is constant we can bound its value, as it is shown by the following.

Proposition 2.8 ([66]). *Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ such that $\bar{J}\bar{H}$ is normal to $\bar{M}$. Assume that it has non-zero constant mean curvature. We have*

(a) If $\bar{M}$ is proper-biharmonic, then $|\bar{H}|^2 \in (0,1]$.

(b) If $|\bar{H}|^2 = 1$, then $\bar{M}$ is proper-biharmonic if and only if it is pseudo-umbilical and $\nabla^\perp \bar{H} = 0$.

Remark 2.9. We shall see in Proposition 2.32 (a), that the upper bound is reached in the case of curves.

2.3 The Hopf fibration and the biharmonic equation

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ be the natural projection. Then π restricted to the sphere $\mathbb{S}^{2n+1}$ of $\mathbb{C}^{n+1}$ gives rise to the Hopf fibration $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ and if $4c = 4$ then $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ defines a Riemannian submersion. In the sequel we shall look at $\mathbb{S}^{2n+1}$ as a hypersurface of $\mathbb{R}^{2n+2}$ and denote by $\hat{J}$ the complex structure of $\mathbb{R}^{2n+2}$.

Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ and denote by $M := \pi^{-1}(\bar{M})$ the Hopf-tube over $\bar{M}$. If we denote by $\bar{\jmath} : \bar{M} \to \mathbb{C}P^n$ and $\jmath : M \to \mathbb{S}^{2n+1}$ the respective inclusions we have the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \jmath\ } & \mathbb{S}^{2n+1} \\ \big\downarrow & & \big\downarrow \pi \\ \bar{M} & \xrightarrow{\ \bar{\jmath}\ } & \mathbb{C}P^n. \end{array}$$

We shall now find the relation between the bitension field of the inclusion $\bar{\jmath}$ and the bitension field of the inclusion $\jmath$. For this, let $\{\bar{X}_k\}_{k=1}^{\bar{m}}$ be a local orthonormal frame field tangent to $\bar{M}$, $1 \leq \bar{m} \leq 2n-1$, and let $\{\bar{\eta}_\alpha\}_{\alpha=\bar{m}+1}^{2n}$ be a local orthonormal frame field normal to $\bar{M}$. Let us denote by $X_k := \bar{X}_k^H$ and $\eta_\alpha := \bar{\eta}_\alpha^H$ the horizontal lifts with respect to the Hopf map and by ξ the Hopf vector field on $\mathbb{S}^{2n+1}$ which is tangent to the fibres of the Hopf fibration, i.e. $\xi(p) = -\hat{J}p$, for any $p \in \mathbb{S}^{2n+1}$. Then $\{\xi, X_k\}$ is a local orthonormal frame field tangent to M and $\{\eta_\alpha\}$ is a local orthonormal frame field normal to M.

Lemma 2.10 ([66])**.** *Let $X = \bar{X}^H \in C(TM)$, where $\bar{X} \in C(T\bar{M})$, and $V = \bar{V}^H \in C(\jmath^{-1}(T\mathbb{S}^{2n+1}))$, where $\bar{V} \in C((\bar{\jmath})^{-1}(T\mathbb{C}P^n))$. Then*

$$\nabla^{\jmath}_X V = (\nabla^{\bar{\jmath}}_{\bar{X}} \bar{V})^H + \langle V, \hat{J}X\rangle \xi = (\nabla^{\bar{\jmath}}_{\bar{X}} \bar{V})^H + (\langle \bar{V}, \bar{J}\bar{X}\rangle \circ \pi)\xi,$$

where $\nabla^{\jmath}$ and $\nabla^{\bar{\jmath}}$ denote the pull-back connections on $\jmath^{-1}(T\mathbb{S}^{2n+1})$ and $(\bar{\jmath})^{-1}(T\mathbb{C}P^n)$, respectively.

Proof. Decomposing $\nabla^{\jmath}_X V$ in its horizontal and vertical components we have

$$\nabla^{\jmath}_X V = \nabla^{\jmath}_{\bar{X}^H} \bar{V}^H = (\nabla^{\bar{\jmath}}_{\bar{X}} \bar{V})^H + \langle \nabla^{\jmath}_X V, \xi\rangle \xi.$$

Now,

$$\begin{aligned} \langle \nabla^{\jmath}_X V, \xi \rangle &= -\langle V, \nabla^{\jmath}_X \xi\rangle = -\langle V, \hat{\nabla}_X \xi + \langle X, \xi\rangle p\rangle \\ &= \langle V, \hat{\nabla}_X \hat{J}p\rangle = \langle V, \hat{J}X\rangle = \langle \bar{V}, \bar{J}\bar{X}\rangle \circ \pi, \end{aligned}$$

where $\hat{\nabla}$ is the Levi-Civita connection on the Euclidean space $\mathbb{R}^{2n+2}$. □

Lemma 2.11 ([66])**.** *If $V = \bar{V}^H \in C(\jmath^{-1}(T\mathbb{S}^{2n+1}))$, $\bar{V} \in C((\bar{\jmath})^{-1}(T\mathbb{C}P^n))$, then*

$$\Delta^{\jmath} V = (\Delta^{\bar{\jmath}} \bar{V})^H + 2\operatorname{div}((\hat{J}V)^\top)\xi + \langle V, \hat{J}\tau(\jmath)\rangle \xi + V - \hat{J}(\hat{J}V)^\top,$$

where $\Delta^{\jmath}$ and $\Delta^{\bar{\jmath}}$ are the rough Laplacians acting on sections of $\jmath^{-1}(T\mathbb{S}^{2n+1})$ and $(\bar{\jmath})^{-1}(T\mathbb{C}P^n)$, respectively, whilst $(V)^\top$ denotes the component of V tangent to M.

Proof. The Laplacian $\Delta^\jmath$ is given by

$$-\Delta^\jmath V = \sum_{i=1}^{\bar{m}} \{\nabla^\jmath_{X_i}\nabla^\jmath_{X_i} V - \nabla^\jmath_{\nabla^M_{X_i}X_i} V\} + \nabla^\jmath_\xi \nabla^\jmath_\xi V - \nabla^\jmath_{\nabla^M_\xi \xi} V.$$

We compute each term separately. From Lemma 2.10 we have

$$\begin{aligned} \nabla^\jmath_{X_i}\nabla^\jmath_{X_i} V &= (\nabla^{\bar{\jmath}}_{\bar{X}_i}\nabla^{\bar{\jmath}}_{\bar{X}_i}\bar{V})^H + \langle \nabla^\jmath_{X_i} V, \hat{J}X_i\rangle\xi + \nabla^\jmath_{X_i}(\langle V, \hat{J}X_i\rangle\xi) \\ &= (\nabla^{\bar{\jmath}}_{\bar{X}_i}\nabla^{\bar{\jmath}}_{\bar{X}_i}\bar{V})^H + 2\langle \nabla^\jmath_{X_i} V, \hat{J}X_i\rangle\xi \\ &\quad + \langle V, \nabla^\jmath_{X_i}\hat{J}X_i\rangle\xi + \langle \hat{J}V, X_i\rangle \hat{J}X_i. \end{aligned}$$

Using

$$\nabla^\jmath_{X_i}\hat{J}X_i = \hat{J}\nabla^\jmath_{X_i}X_i + \xi$$

we get

$$\begin{aligned} \nabla^\jmath_{X_i}\nabla^\jmath_{X_i} V &= (\nabla^{\bar{\jmath}}_{\bar{X}_i}\nabla^{\bar{\jmath}}_{\bar{X}_i}\bar{V})^H + 2\langle \nabla^\jmath_{X_i} V, \hat{J}X_i\rangle\xi \\ &\quad + \langle V, \hat{J}\nabla^\jmath_{X_i}X_i\rangle\xi + \hat{J}(\langle \hat{J}V, X_i\rangle X_i). \end{aligned} \tag{2.8}$$

Next

$$\nabla^\jmath_{\nabla^M_{X_i}X_i} V = (\nabla^{\bar{\jmath}}_{\nabla^{\bar{M}}_{\bar{X}_i}\bar{X}_i}\bar{V})^H + \langle V, \hat{J}\nabla^M_{X_i}X_i\rangle\xi. \tag{2.9}$$

Summing (2.8) and (2.9) up we find

$$\begin{aligned} -\Delta^\jmath V &= -(\Delta^{\bar{\jmath}}\bar{V})^H + 2\sum_{i=1}^{\bar{m}}\langle \nabla^\jmath_{X_i} V, \hat{J}X_i\rangle\xi + \langle V, \hat{J}\sum_{i=1}^{\bar{m}}(\nabla^\jmath_{X_i}X_i - \nabla^M_{X_i}X_i)\rangle\xi \\ &\quad + \sum_{i=1}^{\bar{m}} \hat{J}(\langle \hat{J}V, X_i\rangle X_i) + \nabla^\jmath_\xi\nabla^\jmath_\xi V \\ &= -(\Delta^{\bar{\jmath}}\bar{V})^H + 2\sum_{i=1}^{\bar{m}}\langle \nabla^\jmath_{X_i} V, \hat{J}X_i\rangle\xi + \langle V, \hat{J}\tau(\jmath)\rangle\xi \\ &\quad + \hat{J}(\hat{J}V)^\top + \nabla^\jmath_\xi\nabla^\jmath_\xi V. \end{aligned}$$

We now compute the extra terms in the above equation.

$$\begin{aligned} \sum_{i=1}^{\bar{m}}\langle \nabla^\jmath_{X_i} V, \hat{J}X_i\rangle &= \sum_{i=1}^{\bar{m}}\{-X_i\langle \hat{J}V, X_i\rangle + \langle \hat{J}V, \nabla^\jmath_{X_i}X_i\rangle\} \\ &= \langle \hat{J}V, \tau(\jmath)\rangle - \sum_{i=1}^{\bar{m}}\{X_i\langle \hat{J}V, X_i\rangle - \langle \hat{J}V, \nabla^M_{X_i}X_i\rangle\} \\ &= \langle \hat{J}V, \tau(\jmath)\rangle - \operatorname{div}((\hat{J}V)^\top). \end{aligned} \tag{2.10}$$

Finally

$$\begin{aligned} \nabla^\jmath_\xi V &= H(\nabla^\jmath_\xi V) + \langle \nabla^\jmath_\xi V, \xi\rangle\xi = H(\nabla^\jmath_\xi V) \\ &= H(\nabla^\jmath_V \xi) = H(\hat{\nabla}_V\xi + \langle V, \xi\rangle p) = H(-\hat{J}V) = -\hat{J}V \end{aligned}$$

which gives

$$\nabla^{\jmath}_{\xi}\nabla^{\jmath}_{\xi}V = -V.$$

□

Before giving the relation between the bitension fields we need to compute the trace of the curvature operators. One gets immediately

$$-\operatorname{trace} R^{\mathbb{S}^{2n+1}}(d\jmath, \tau(\jmath))d\jmath = (\bar{m}+1)\tau(\jmath) \tag{2.11}$$

and

$$-\operatorname{trace} R^{\mathbb{C}P^n}(d\bar{\jmath}, \tau(\bar{\jmath}))d\bar{\jmath} = \bar{m}\tau(\bar{\jmath}) - 3\bar{J}(\bar{J}\tau(\bar{\jmath}))^{\top}. \tag{2.12}$$

We are now ready to state the main theorem of this section.

Theorem 2.12 ([66]). *Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ of dimension $\bar{m}$ and denote by $M := \pi^{-1}(\bar{M})$ the corresponding Hopf-tube. If we denote by $\bar{\jmath} : \bar{M} \to \mathbb{C}P^n$ and $\jmath : M \to \mathbb{S}^{2n+1}$ the respective inclusions we have that*

$$(\tau_2(\bar{\jmath}))^H = \tau_2(\jmath) - 4\hat{J}(\hat{J}\tau(\jmath))^{\top} + 2\operatorname{div}((\hat{J}\tau(\jmath))^{\top})\xi. \tag{2.13}$$

Proof. From (0.1) and (2.11) we have

$$\tau_2(\jmath) = -\Delta^{\jmath}\tau(\jmath) + (\bar{m}+1)\tau(\jmath).$$

Next, since $\tau(\jmath) = (\tau(\bar{\jmath}))^H$, using Lemma 2.11 and (2.12) we find the assertion of the theorem. □

Remark 2.13 ([66]). (i) Using the horizontal lift, it is straightforward to check that (2.13) can be written as

$$(\tau_2(\bar{\jmath}))^H = \tau_2(\jmath) - 4(\bar{J}(\bar{J}\tau(\bar{\jmath}))^{\top})^H + 2(\operatorname{div}_{\bar{M}}((\bar{J}\tau(\bar{\jmath}))^{\top}) \circ \pi)\xi.$$

(ii) If $\hat{J}\tau(\jmath)$ is normal to M, then $\tau_2(\bar{\jmath}) = 0$ if and only if $\tau_2(\jmath) = 0$.

(iii) If $\hat{J}\tau(\jmath)$ is tangent to M, then $\tau_2(\bar{\jmath}) = 0$ and $\operatorname{div}_{\bar{M}}((\bar{J}\tau(\bar{\jmath}))^{\top}) = 0$ if and only if $\tau_2(\jmath) + 4\tau(\jmath) = 0$.

(iv) Assume that, locally, $M = \pi^{-1}(\bar{M}) = \mathbb{S}^1 \times \tilde{M}$, where $\tilde{M}$ is an integral submanifold of $\mathbb{S}^{2n+1}$, i.e. $\langle \tilde{X}_{\tilde{p}}, \xi(\tilde{p})\rangle = 0$, for any vector $\tilde{X}_{\tilde{p}}$ tangent to $\tilde{M}$. Denote by $\tilde{\jmath} : \tilde{M} \to \mathbb{S}^{2n+1}$ the canonical inclusion, and by $\{\phi_t\}$ the flow of ξ. We know that $\tau_2(\jmath)_{(t,\tilde{p})} = (d\phi_t)_{\tilde{p}}(\tau_2(\tilde{\jmath}))$, see [70], and we can check that, at $\tilde{p}$,

$$(\tau_2(\bar{\jmath}))^H = \tau_2(\tilde{\jmath}) - 4\hat{J}(\hat{J}\tau(\tilde{\jmath}))^{\top} + 2\operatorname{div}_{\tilde{M}}((\hat{J}\tau(\tilde{\jmath}))^{\top})\xi.$$

To state the next results we recall that a smooth map $\varphi : (M, g) \to (N, h)$ is called λ-*biharmonic* if it is a critical point of the λ-*bienergy*

$$E_2(\varphi) + \lambda E(\varphi),$$

where λ is a real constant. The critical points of the λ-bienergy satisfy the equation

$$\tau_2(\varphi) - \lambda\tau(\varphi) = 0.$$

Proposition 2.14 ([66]). *Let $\bar{M}$ be a real hypersurface of $\mathbb{C}P^n$ of constant mean curvature and denote by $M = \pi^{-1}(\bar{M})$ the Hopf-tube over $\bar{M}$. Then $\tau_2(\bar{\jmath}) = 0$ if and only if $\tau_2(\jmath) + 4\tau(\jmath) = 0$, i.e. $\jmath$ is (-4)-biharmonic.*

Proof. We have $(\bar{J}\tau(\bar{\jmath}))^\top = \bar{J}\tau(\bar{\jmath})$ and it remains to prove that $\operatorname{div}_{\bar{M}}(\bar{J}\tau(\bar{\jmath})) = 0$. Let $\bar{\eta}$ be a local unit section in the normal bundle of $\bar{M}$ in $\mathbb{C}P^n$ and consider $\{\bar{X}_1, \bar{J}\bar{X}_1, \ldots, \bar{X}_{n-1}, \bar{J}\bar{X}_{n-1}, \bar{J}\bar{\eta}\}$ a local orthonormal frame field tangent to $\bar{M}$. Since $\bar{M}$ is a hypersurface of constant mean curvature, it is enough to prove that $\operatorname{div}_{\bar{M}}(\bar{J}\bar{\eta}) = 0$. But, denoting by $\bar{A}_{\bar{\eta}}$ the shape operator of $\bar{M}$,

$$\langle \nabla^{\bar{M}}_{\bar{X}_a} \bar{J}\bar{\eta}, \bar{X}_a\rangle = \langle \bar{A}_{\bar{\eta}}(\bar{X}_a), \bar{J}\bar{X}_a\rangle, \quad \langle \nabla^{\bar{M}}_{\bar{J}\bar{X}_b} \bar{J}\bar{\eta}, \bar{J}\bar{X}_b\rangle = -\langle \bar{A}_{\bar{\eta}}(\bar{X}_b), \bar{J}\bar{X}_b\rangle,$$

for any $1 \leq a, b \leq n-1$, and

$$\langle \nabla^{\bar{M}}_{\bar{J}\bar{\eta}} \bar{J}\bar{\eta}, \bar{J}\bar{\eta}\rangle = 0,$$

so we conclude. □

Proposition 2.15 ([66]). *Let $\bar{M}$ be a Lagrangian submanifold of $\mathbb{C}P^n$ with parallel mean curvature vector field and denote by $M = \pi^{-1}(\bar{M})$ the Hopf-tube over $\bar{M}$. Then $\bar{\jmath}$ is biharmonic if and only if $\jmath$ is (-4)-biharmonic.*

Proof. Since $\bar{M}$ is a Lagrangian submanifold, $\dim \bar{M} = \bar{m} = n$ and $\bar{J}(T\bar{M}) = N\bar{M}$ (therefore $\bar{J}(N\bar{M}) = T\bar{M}$). We have that $\bar{J}\tau(\bar{\jmath}) \in C(T\bar{M})$ and we shall prove that $\nabla^{\bar{M}}\bar{J}\tau(\bar{\jmath}) = 0$ which implies $\operatorname{div}_{\bar{M}}(\bar{J}\tau(\bar{\jmath})) = 0$. Indeed, for any vector fields $\bar{X}$ and $\bar{Y}$ tangent to $\bar{M}$ we have

$$\begin{aligned}\langle \nabla^{\bar{M}}_{\bar{X}} \bar{J}\tau(\bar{\jmath}), \bar{Y}\rangle &= \langle \nabla^{\bar{\jmath}}_{\bar{X}} \bar{J}\tau(\bar{\jmath}), \bar{Y}\rangle = \langle \bar{J}\nabla^{\bar{\jmath}}_{\bar{X}}\tau(\bar{\jmath}), \bar{Y}\rangle = \langle -\bar{J}\bar{A}_{\tau(\bar{\jmath})}(\bar{X}), \bar{Y}\rangle \\ &= 0.\end{aligned}$$

□

We end this section with the following result.

Proposition 2.16 ([66]). *Let $\bar{M}$ be a real submanifold of $\mathbb{C}P^n$ such that $\bar{J}\tau(\bar{\jmath})$ is normal to $\bar{M}$ and denote by $M = \pi^{-1}(\bar{M})$ the Hopf-tube over $\bar{M}$. Then $\bar{\jmath}$ is biharmonic if and only if $\jmath$ is biharmonic.*

2.4 Biharmonic submanifolds of Clifford type

For a fixed $n > 1$, consider the spheres $\mathbb{S}^{2p+1}(a) \subset \mathbb{R}^{2p+2} = \mathbb{C}^{p+1}$ and $\mathbb{S}^{2q+1}(b) \subset \mathbb{R}^{2q+2} = \mathbb{C}^{q+1}$, with $a^2+b^2 = 1$ and $p+q = n-1$. Denote by $T^{p,q}_{a,b} = \mathbb{S}^{2p+1}(a)\times\mathbb{S}^{2q+1}(b) \subset \mathbb{S}^{2n+1}$ the Clifford torus. Let now M_1 be a minimal submanifold of $\mathbb{S}^{2p+1}(a)$ of dimension m_1 and M_2 a minimal submanifold of $\mathbb{S}^{2q+1}(b)$ of dimension m_2. The submanifold $M_1 \times M_2$ is clearly minimal in $T^{p,q}_{a,b}$ and, according to [29], is proper-biharmonic in $\mathbb{S}^{2n+1}$ if and only if $a = b = \sqrt{2}/2$ and $m_1 \neq m_2$. If $M_1 \times M_2$ is invariant under the action of the one-parameter group of isometries generated by the Hopf vector field ξ on $\mathbb{S}^{2n+1}$, then it projects onto a submanifold of $\mathbb{C}P^n$ and we could ask for which values of a, b, m_1, m_2 is it a proper-biharmonic submanifold.

We start with the following lemma.

Lemma 2.17 ([66]). *Let denote by $\jmath_1 : M_1^{m_1} \times M_2^{m_2} \to T^{p,q}_{a,b}$ the inclusion of $M_1 \times M_2$ in the Clifford torus and by $\jmath : T^{p,q}_{a,b} \to \mathbb{S}^{2n+1}$ the inclusion of the Clifford torus in the sphere. Then*

$$\begin{cases} \tau(\jmath \circ \jmath_1) = (\frac{a}{b} m_2 - \frac{b}{a} m_1)\eta = c\eta \\ \tau_2(\jmath \circ \jmath_1) = c(m_1 + m_2 - \frac{b^2}{a^2} m_1 - \frac{a^2}{b^2} m_2)\eta \end{cases}, \tag{2.14}$$

where η is the unit normal section in the normal bundle of $T^{p,q}_{a,b}$ in $\mathbb{S}^{2n+1}$ given by $\eta(x,y) = (\frac{b}{a}x, -\frac{a}{b}y)$, $x \in \mathbb{S}^{2p+1}(a), y \in \mathbb{S}^{2q+1}(b)$.

Proof. Let $p = (x,y) \in T^{p,q}_{a,b}$, $x \in \mathbb{R}^{2p+2}$, $y \in \mathbb{R}^{2q+2}$, $|x| = a$, $|y| = b$. Then $\eta(x,y) = (\frac{b}{a}x, -\frac{a}{b}y)$ defines a unit normal section in the normal bundle of $T^{p,q}_{a,b}$ in $\mathbb{S}^{2n+1}$. We identify $X = (X,0) \in T_pT^{p,q}_{a,b}$, $Y = (0,Y) \in T_pT^{p,q}_{a,b}$, and a straightforward computation gives

$$\nabla^{\jmath}_X \eta = -A^{\jmath}(X) = \frac{b}{a}X, \quad \nabla^{\jmath}_Y \eta = -A^{\jmath}(Y) = -\frac{a}{b}Y.$$

Let $\{X_k = (X_k, 0)\}$ be a local orthonormal frame field tangent to $\mathbb{S}^{2p+1}(a)$ and $\{Y_l = (0, Y_l)\}$ a local orthonormal frame field tangent to $\mathbb{S}^{2q+1}(b)$. Then, applying the composition law for the tension field and using that $\jmath_1$ is harmonic, we have

$$\begin{aligned} \tau(\jmath \circ \jmath_1) &= d\jmath(\tau(\jmath_1)) + \operatorname{trace} \nabla d\jmath(d\jmath_1, d\jmath_1) \\ &= \sum_{k=1}^{m_1} \langle A^{\jmath}(X_k), X_k\rangle\eta + \sum_{l=1}^{m_2} \langle A^{\jmath}(Y_l), Y_l\rangle\eta = (\frac{a}{b}m_2 - \frac{b}{a}m_1)\eta = c\,\eta. \end{aligned}$$

To compute $\tau_2(\jmath \circ \jmath_1)$, let us choose around $p = (x,y) \in M_1 \times M_2$ a frame field $\{(X_k, Y_l)\}$ such that $\{X_k\}_{k=1}^{m_1}$ is a geodesic frame field around x and $\{Y_l\}_{l=1}^{m_2}$ is a geodesic frame field around y. Then at p

$$\begin{aligned} -\Delta^{\jmath\circ\jmath_1}\eta &= \sum_{k=1}^{m_1} \nabla^{\jmath\circ\jmath_1}_{X_k} \nabla^{\jmath\circ\jmath_1}_{X_k} \eta + \sum_{l=1}^{m_2} \nabla^{\jmath\circ\jmath_1}_{Y_l} \nabla^{\jmath\circ\jmath_1}_{Y_l} \eta \\ &= \frac{b}{a} \sum_{k=1}^{m_1} \nabla^{\jmath\circ\jmath_1}_{X_k} X_k - \frac{a}{b} \sum_{l=1}^{m_2} \nabla^{\jmath\circ\jmath_1}_{Y_l} Y_l \\ &= \frac{b}{a} \sum_{k=1}^{m_1} (B^{\jmath}(X_k, X_k) + \nabla^{T^{p,q}_{a,b}}_{X_k} X_k) - \frac{a}{b} \sum_{l=1}^{m_2} (B^{\jmath}(Y_l, Y_l) + \nabla^{T^{p,q}_{a,b}}_{Y_l} Y_l) \quad (2.15) \\ &= \frac{b}{a} \sum_{k=1}^{m_1} B^{\jmath}(X_k, X_k) - \frac{a}{b} \sum_{l=1}^{m_2} B^{\jmath}(Y_l, Y_l) \\ &= (-\frac{b^2}{a^2} m_1 - \frac{a^2}{b^2} m_2)\eta. \end{aligned}$$

Finally, using the standard formula for the curvature of $\mathbb{S}^{2n+1}$, we get

$$-\operatorname{trace} R^{\mathbb{S}^{2n+1}}(d(\jmath \circ \jmath_1), \tau(\jmath \circ \jmath_1))d(\jmath \circ \jmath_1) = (m_1 + m_2)\tau(\jmath \circ \jmath_1) = (m_1 + m_2)c\eta,$$

that summed up with (2.15) gives the lemma. □

Theorem 2.18 ([66]). *Let $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ be the Hopf map. Let $M = M_1^{m_1} \times M_2^{m_2}$ be the product of two minimal submanifolds of $\mathbb{S}^{2p+1}(a)$ and $\mathbb{S}^{2q+1}(b)$, respectively. Assume that M is invariant under the action of the one-parameter group of isometries generated by the Hopf vector field ξ on $\mathbb{S}^{2n+1}$. Then $\pi(M)$ is a proper-biharmonic submanifold of $\mathbb{C}P^n$ if and only if M is (-4)-biharmonic, that is*

$$\begin{cases} a^2 + b^2 = 1 \\ \frac{a}{b} m_2 - \frac{b}{a} m_1 \neq 0 \\ \frac{b^2}{a^2} m_1 + \frac{a^2}{b^2} m_2 = 4 + m_1 + m_2 \end{cases}, \tag{2.16}$$

where m_1 and m_2 are the dimensions of M_1 and M_2, respectively.

Proof. The Hopf vector field ξ is a Killing vector field on $\mathbb{S}^{2n+1}$ that, at a point $p = (x, y)$, is given by

$$\xi = -(-x^2, x^1, \dots, -x^{2p+2}, x^{2p+1}, -y^2, y^1, \dots, -y^{2q+2}, y^{2q+1}) = (\xi_1, \xi_2).$$

Since $M_1 \times M_2$ is invariant under the action of the one-parameter group of isometries generated by ξ, it remains Killing when restricted to $M_1 \times M_2$. As

$$\hat{J}\eta = (-\frac{b}{a}\xi_1, \frac{a}{b}\xi_2),$$

it follows that $\hat{J}\eta$ is a Killing vector field on $M_1 \times M_2$.

Since $\operatorname{div}(\hat{J}\tau(\jmath \circ \jmath_1)) = \operatorname{div}(c\hat{J}\eta) = 0$, using Remark 2.13 (iii), it results that $\pi(M_1 \times M_2)$ is a biharmonic submanifold of $\mathbb{C}P^n$ if and only if

$$\tau_2(\jmath \circ \jmath_1) + 4\tau(\jmath \circ \jmath_1) = 0.$$

Finally, using Lemma 2.17, we get

$$\tau_2(\jmath \circ \jmath_1) + 4\tau(\jmath \circ \jmath_1) = c(4 + m_1 + m_2 - \frac{b^2}{a^2} m_1 - \frac{a^2}{b^2} m_2)\eta.$$

□

Remark 2.19 ([66]). If $M_1 = \mathbb{S}^{2p+1}(a)$ and $M_2 = \mathbb{S}^{2q+1}(b)$, we recover the result in [77, 78] concerning the proper-biharmonic homogeneous real hypersurfaces of type A in $\mathbb{C}P^n$.

Example 2.20 ([66]). Let e_1 and e_3 be two constant unit vectors in $\mathbb{E}^{2n+2}$, with e_3 orthogonal to e_1 and $\hat{J}e_1$. We consider the circles $\mathbb{S}^1(a)$ and $\mathbb{S}^1(b)$ lying in the 2-planes spanned by $\{e_1, \hat{J}e_1\}$ and $\{e_3, \hat{J}e_3\}$, respectively. Then $M = \mathbb{S}^1(a) \times \mathbb{S}^1(b)$ is invariant under the flow-action of ξ, and $\pi(M)$ is a proper-biharmonic curve of $\mathbb{C}P^n$ if and only if $a = \frac{\sqrt{2 \pm \sqrt{2}}}{2}$.

Example 2.21 ([66]). For $p = 0$ and $q = n-1$, we get that $\pi(\mathbb{S}^1(a) \times \mathbb{S}^{2n-1}(b))$ is proper-biharmonic in $\mathbb{C}P^n$ if and only if $a^2 = \frac{n+3 \pm \sqrt{n^2+2n+5}}{4(n+1)}$. In particular, $\pi(\mathbb{S}^1(a) \times \mathbb{S}^3(b))$ is a proper-biharmonic real hypersurface in $\mathbb{C}P^2$ if and only if $a^2 = \frac{5 \pm \sqrt{13}}{12}$.

Example 2.22 ([66])**.** If $p = q$ then $M = T^{p,p}_{a,b}$ is never a proper-biharmonic hypersurface of $\mathbb{S}^{2n+1}$, and it is easy to check that $\pi(M)$ is a proper-biharmonic hypersurface of $\mathbb{C}P^n$ if and only if $a^2 = \frac{2p+2-\sqrt{2(p+1)}}{4(p+1)}$.

Example 2.23 ([66])**.** Let $M = \mathbb{S}^{2p+1}(a) \times \mathbb{S}^p\left(\frac{b}{\sqrt{2}}\right) \times \mathbb{S}^p\left(\frac{b}{\sqrt{2}}\right)$, p odd. Then M is minimal in $T^{p,p}_{a,b}$, and is proper-biharmonic in $\mathbb{S}^{2n+1}$ if and only if $a = b = \frac{1}{\sqrt{2}}$. By a straightforward computation we can check that $\pi(M)$ is proper-biharmonic in $\mathbb{C}P^n$ if and only if $a^2 = \frac{8p+7\pm\sqrt{32p+25}}{16p+12}$.

2.4.1 Sphere bundle of all vectors tangent to $\mathbb{S}^{2p+1}(a)$

We have seen that if M is a product submanifold in $T^{p,q}_{a,b}$ then its projection $\pi(M)$ can be proper-biharmonic in $\mathbb{C}P^n$. But when M is not a product, the situation can be more complicated as it is illustrated by the following example.

We consider the sphere of radius a

$$\mathbb{S}^{2p+1}(a) = \{x \in \mathbb{R}^{2p+2} : (x^1)^2 + \cdots + (x^{2p+2})^2 = a^2\}$$

and its sphere bundle of all vectors tangent to $\mathbb{S}^{2p+1}(a)$ and of norm b, that is

$$M = T^b\mathbb{S}^{2p+1}(a) = \{(x, y) \in \mathbb{R}^{4p+4} : x, y \in \mathbb{R}^{2n+2}, |x| = a, |y| = b, \langle x, y\rangle = 0\}.$$

It is easy to check that M is invariant under the flow-action of the characteristic vector field ξ, which means $e^{-\mathrm{i}t}p \in M$, $\forall p \in M$ and $\forall t \in \mathbb{R}$. Let $(x_0, y_0) \in M$. Then

$$\begin{aligned} T_{(x_0,y_0)}M = \{Z_0 = (X_0, Y_0) \in \mathbb{R}^{4p+4} \quad : \quad & \langle x_0, X_0\rangle = 0,\ \langle y_0, Y_0\rangle = 0, \\ & \langle X_0, y_0\rangle + \langle x_0, Y_0\rangle = 0\}. \end{aligned}$$

In order to find a basis in $T_{(x_0,y_0)}M$, we consider $\{y_0, y_2, y_3, \dots, y_{2p+1}\}$ an orthogonal basis in $T_{x_0}\mathbb{S}^{2p+1}(a)$, each vector being of norm b. We think M as a hypersurface of the tangent bundle $T\mathbb{S}^{2p+1}(a)$, and we consider on $T\mathbb{S}^{2p+1}(a)$ and M the induced metrics from the canonical metric on $\mathbb{R}^{4p+4}$

$$M \to T\mathbb{S}^{2p+1}(a) \to \mathbb{R}^{4p+4}.$$

The above inclusions are the canonical ones.

The vertical lifts of the tangent vectors $y_2, y_3, \dots, y_{2p+1}$, in (x_0, y_0), are

$$y_2^V = (0, y_2),\ y_3^V = (0, y_3), \dots,\ y_{2p+1}^V = (0, y_{2p+1}),$$

and the horizontal lifts of $y_0, y_2, y_3, \dots, y_{2p+1}$, in (x_0, y_0), are

$$y_0^H = (y_0, -\frac{b^2}{a^2}x_0),\ y_2^H = (y_2, 0),\ y_3^H = (y_3, 0), \dots,\ y_{2p+1}^H = (y_{2p+1}, 0).$$

The vectors $\{y_0^H, y_2^H, y_3^H, \dots, y_{2p+1}^H, y_2^V, y_3^V, \dots, y_{2p+1}^V\}$ form an orthogonal basis in $T_{(x_0,y_0)}M$ and

$$|y_2^V| = |y_3^V| = \cdots = |y_{2p+1}^V| = b,\ |y_2^H| = |y_3^H| = \cdots = |y_{2p+1}^H| = b,\ |y_0^H| = \frac{b}{a}.$$

The vector $C(x_0, y_0) = y_0^V = (0, y_0)$ is tangent to $T\mathbb{S}^{2p+1}(a)$ in (x_0, y_0) and orthogonal to M.

From now on we shall consider $a^2 + b^2 = 1$ and the inclusions

$$M \to \mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b) \to \mathbb{S}^{4p+3} \to \mathbb{R}^{4p+4}.$$

We define $\eta_1(x_0, y_0) = (y_0, x_0)$ and $\eta_2(x_0, y_0) = (x_0, -\frac{a^2}{b^2} y_0)$. We have that η_1 and η_2 are normal to M, and

$$\eta_1(x_0, y_0) \in T_{(x_0,y_0)}(\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)), \quad |\eta_1(x_0, y_0)| = 1$$

$$\eta_2(x_0, y_0) \in T_{(x_0,y_0)}\mathbb{S}^{4p+3},\ \eta_2(x_0, y_0) \perp T_{(x_0,y_0)}(\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)),\ |\eta_2(x_0, y_0)| = \frac{a}{b}.$$

We denote by $B_{(x_0,y_0)}$ the second fundamental form of M in $\mathbb{S}^{4p+3}$, in the point (x_0, y_0). By a straightforward computation we obtain

$$B_{(x_0,y_0)}(Z_0, Z_0) = -2\langle X_0, Y_0\rangle \eta_1 - \frac{b^2}{a^2}(|X_0|^2 - \frac{a^2}{b^2}|Y_0|^2)\eta_2, \tag{2.17}$$

where $Z_0 = (X_0, Y_0) \in T_{(x_0,y_0)}M$. From (2.17) we get

$$H(x_0, y_0) = \frac{2p}{4p+1}\frac{a^2 - b^2}{a^2}\eta_2 = c\eta_2.$$

Therefore M is minimal in $\mathbb{S}^{4p+3}$ if and only if $a = b = \frac{1}{\sqrt{2}}$.

It is not difficult to check that

$$\begin{cases} \nabla^{\mathbb{S}^{4p+3}}_{y_0^H}\eta_2 = \eta_1,\ \nabla^{\mathbb{S}^{4p+3}}_{y_2^H}\eta_2 = y_2^H,\ \nabla^{\mathbb{S}^{4p+3}}_{y_3^H}\eta_2 = y_3^H, \ldots,\ \nabla^{\mathbb{S}^{4p+3}}_{y_{2p+1}^H}\eta_2 = y_{2p+1}^H \\ \nabla^{\mathbb{S}^{4p+3}}_{y_2^V}\eta_2 = -\frac{a^2}{b^2}y_2^V,\ \nabla^{\mathbb{S}^{4p+3}}_{y_3^V}\eta_2 = -\frac{a^2}{b^2}y_3^V, \ldots,\ \nabla^{\mathbb{S}^{4p+3}}_{y_{2p+1}^V}\eta_2 = -\frac{a^2}{b^2}y_{2p+1}^V \\ \nabla^{\mathbb{S}^{4p+3}}_{y_0^H}\eta_1 = -\frac{b^2}{a^2}\eta_2,\ \nabla^{\mathbb{S}^{4p+3}}_{y_2^H}\eta_1 = y_2^V,\ \nabla^{\mathbb{S}^{4p+3}}_{y_3^H}\eta_1 = y_3^V, \ldots,\ \nabla^{\mathbb{S}^{4p+3}}_{y_{2p+1}^H}\eta_1 = y_{2p+1}^V \\ \nabla^{\mathbb{S}^{4p+3}}_{y_2^V}\eta_1 = y_2^H,\ \nabla^{\mathbb{S}^{4p+3}}_{y_3^V}\eta_1 = y_3^H, \ldots,\ \nabla^{\mathbb{S}^{4p+3}}_{y_{2p+1}^V}\eta_1 = y_{2p+1}^H \end{cases}. \tag{2.18}$$

From (2.18) we obtain that

$$\operatorname{trace} A_{\nabla^\perp_{(\cdot)}\eta_2}(\cdot) = 0 \quad \text{and} \quad \operatorname{trace} B(\cdot, A_{\eta_2}(\cdot)) = 2p(\frac{a^2}{b^2} + \frac{b^2}{a^2})\eta_2. \tag{2.19}$$

Denoting $W(x_0, y_0) = y_0^H$, we get

$$-\Delta^\perp\eta_2 = \frac{a^2}{b^2}(\nabla^\perp_W\nabla^\perp_W\eta_2 - \nabla^\perp_{\nabla^M_W W}\eta_2) = -\eta_2. \tag{2.20}$$

Before concluding we give the following Lemma which follows by direct computation.

Lemma 2.24 ([66]). *Let N^n be a hypersurface of a Riemmanian manifold $(P^{n+1}, \langle,\rangle)$, and $X \in C(TP)$ a Killing vector field. We denote $X^\top = (X_{/N})^\top \in C(TN)$. Then* $\operatorname{div} X^\top = n\langle H, X\rangle$*, where H is the mean curvature vector field of N. In particular, if N is minimal then* $\operatorname{div} X^\top = 0$.

Now we can state

Proposition 2.25 ([66]). *Let $M = T^b\mathbb{S}^{2p+1}(a)$ be the sphere bundle of all vectors of norm b tangent to $\mathbb{S}^{2p+1}(a)$. Assume that $a^2 + b^2 = 1$ and $p \geq 1$. Then we have*

(a) M is never proper-biharmonic in $\mathbb{S}^{4p+3}$.

(b) M is (-4)-biharmonic in $\mathbb{S}^{4p+3}$ if and only if $a^2 = \frac{2p+1\pm\sqrt{2p+1}}{4p+2}$.

(c) M is minimal in $T^{p,p}_{a,b} = \mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)$.

(d) $\pi(M)$ is never proper-biharmonic in $\mathbb{C}P^n$.

Proof. As the mean curvature vector field of M in $\mathbb{S}^{4p+3}$ is $H = c\eta_2$, where $c = \frac{2p}{4p+1}\frac{a^2-b^2}{a^2}$, then M is biharmonic if and only if

$$\begin{cases} -\Delta^\perp\eta_2 - \operatorname{trace} B(\cdot, A_{\eta_2}(\cdot)) + (4p+1)\eta_2 = 0 \\ 2\operatorname{trace} A_{\nabla^\perp_{(\cdot)}\eta_2}(\cdot) + \frac{4p+1}{2}\operatorname{grad}(c|\eta_2|^2) = 0 \end{cases}. \tag{2.21}$$

From (2.19) and (2.20) we get that M is biharmonic if and only if

$$-\eta_2 - 2p(\frac{a^2}{b^2} + \frac{b^2}{a^2})\eta_2 + (4p+1)\eta_2 = 0,$$

which is equivalent to $a = b$, that is M is minimal in $\mathbb{S}^{4p+3}$.

(b) We obtain that M is (-4)-biharmonic if and only if

$$-\eta_2 - 2p(\frac{a^2}{b^2} + \frac{b^2}{a^2})\eta_2 + (4p+1)\eta_2 + 4\eta_2 = 0,$$

which holds if and only if $a^2 = \frac{2p+1\pm\sqrt{2p+1}}{4p+2}$.

(c) We denote by $\dot{A}$ the shape operator of M in $\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)$, $\dot{A} = \dot{A}_{\eta_1}$. We can check that

$$\begin{cases} \dot{A}(y_0^H) = 0,\ \dot{A}(y_2^H) = -y_2^V,\ \dot{A}(y_3^H) = -y_3^V, \ldots,\ \dot{A}(y_{2p+1}^H) = -y_{2p+1}^V \\ \dot{A}(y_2^V) = -y_2^H,\ \dot{A}(y_3^V) = -y_3^H, \ldots,\ \dot{A}(y_{2p+1}^V) = -y_{2p+1}^H \end{cases} \tag{2.22}$$

and therefore $\operatorname{trace}\dot{A} = 0$, which means that M is minimal in $\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)$.

(d) We first define

$$\xi_3(x,y) = (\hat{J}x, -\frac{a^2}{b^2}\hat{J}y) = (-\xi_1, \frac{a^2}{b^2}\xi_2), \quad \forall (x,y) \in \mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b).$$

The vector field ξ_3 is a Killing vector field on $\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)$. We observe that $\xi_{3/M} = \hat{J}\eta_2$. Since M is minimal in $\mathbb{S}^{2p+1}(a) \times \mathbb{S}^{2p+1}(b)$, from Lemma 2.24, we get $\operatorname{div}(\hat{J}\eta_2)^\top = 0$. Therefore $\pi(M)$ is biharmonic in $\mathbb{C}P^n$ if and only if

$$\tau_2(\jmath) - 4\hat{J}(\hat{J}\tau(\jmath))^\top = 0,$$

which is not satisfied. $\square$

2.4.2 Circles products.

We shall recover a result of Zhang (see [139]).
We denote by $\mathcal{T}$ the $(n+1)$-dimensional Clifford torus

$$\jmath : \mathcal{T} = \mathbb{S}^1(a_1) \times \cdots \times \mathbb{S}^1(a_{n+1}) \to \mathbb{S}^{2n+1},$$

where $a_1^2 + \cdots + a_{n+1}^2 = 1$. The projection $\bar{\mathcal{T}} = \pi(\mathcal{T})$ is a Lagrangian submanifold in $\mathbb{C}P^n$ of parallel mean curvature vector field.

Theorem 2.26 ([139]). *The Lagrangian submanifold $\bar{\mathcal{T}} = \pi(\mathcal{T})$ of $\mathbb{C}P^n$ is proper-biharmonic if and only if $\mathcal{T}$ is (-4)-biharmonic, that is*

$$\begin{cases} a_{k_0}^2 \neq \frac{1}{n+1} & \text{for some } k_0 \in \{1, 2, \ldots, n+1\} \\ d\, a_k - \frac{1}{a_k^3} = \frac{2}{a_k}(n+3)((n+1)a_k^2 - 1), & k \in \{1, 2, \ldots, n+1\} \end{cases}, \tag{2.23}$$

where $d = \sum_{j=1}^{n+1} \frac{1}{a_j^2}$.

Proof. We denote a point $x \in \mathcal{T}$ by

$$x = (x_1, \ldots, x_{n+1}) = (x_1^1, x_1^2, \ldots, x_{n+1}^1, x_{n+1}^2),$$

where we identify

$$x_k = (x_k^1, x_k^2) = (0, 0, \ldots, 0, 0, x_k^1, x_k^2, 0, 0, \ldots, 0, 0), \quad k = 1, \ldots, n+1.$$

We define $\eta_k(x) = \frac{1}{a_k} x_k$ and $X_k = \hat{J}\eta_k$, $k = 1, \ldots, n+1$, where

$$\hat{J}(x_1^1, x_1^2, \ldots, x_{n+1}^1, x_{n+1}^2) = (-x_1^2, x_1^1, \ldots, -x_{n+1}^2, x_{n+1}^1).$$

The vector fields $\{X_k\}$ form an orthonormal frame field of $C(T\mathcal{T})$. It is easy to check that, at a point x,

$$B(X_k, X_k) = -\frac{1}{a_k}\eta_k + x$$

and for $k \neq j$:

$$B(X_k, X_j) = 0.$$

Therefore $\tau(\jmath) = \sum_{k=1}^{n+1}((n+1)a_k - \frac{1}{a_k})\eta_k$, which implies that $(\hat{J}\tau(\jmath))^\top = \hat{J}\tau(\jmath)$ and $\operatorname{div}(\hat{J}\tau(\jmath)) = 0$.
Since $\nabla^\perp \tau(\jmath) = 0$ and $A_{\tau(\jmath)}(X_k) = -((n+1) - \frac{1}{a_k^2})X_k$, by a straightforward computation we get $\tau_2(\jmath) + 4\tau(\jmath) = 0$ if and only if the desired relation is satisfied. $\square$

Remark 2.27 ([66]). Following [139], for $n = 2$, we obtain that $\bar{\mathcal{T}}$ is a proper-biharmonic Lagrangian surface in $\mathbb{C}P^2$ if and only if $a_1^2 = \frac{9 \pm \sqrt{41}}{20}$ and $a_2^2 = a_3^2 = \frac{11 \mp \sqrt{41}}{40}$ (see also [119]).

2.5 Biharmonic curves in $\mathbb{C}P^n$

Let $\bar\gamma : I \subset \mathbb{R} \to \mathbb{C}P^n$ be a curve parametrized by arc-length. The curve γ is called a *Frenet curve of osculating order* d, $1 \leq d \leq 2n$, if there exist d orthonormal vector fields $\{\bar E_1 = \bar\gamma', \ldots, \bar E_d\}$ along $\bar\gamma$ such that

$$\begin{cases} \bar\nabla_{\bar E_1} \bar E_1 = \bar\kappa_1 \bar E_2 \\ \bar\nabla_{\bar E_1} \bar E_i = -\bar\kappa_{i-1} \bar E_{i-1} + \bar\kappa_i \bar E_{i+1}, \quad \forall i = 2, \ldots, d-1 \\ \bar\nabla_{\bar E_1} \bar E_d = -\bar\kappa_{d-1} \bar E_{d-1} \end{cases}, \tag{2.24}$$

where $\{\bar\kappa_1, \bar\kappa_2, \bar\kappa_3, \ldots, \bar\kappa_{d-1}\}$ are positive functions on I called the *curvatures* of $\bar\gamma$ and $\bar\nabla$ denotes the Levi-Civita connection on $\mathbb{C}P^n$.

A Frenet curve of osculating order d is called a *helix of order* d if $\bar\kappa_i = \text{constant} > 0$ for $1 \leq i \leq d-1$. A helix of order 2 is called a *circle*, and a helix of order 3 is simply called *helix*.

Following S. Maeda and Y. Ohnita [92], we define the *complex torsions* of the curve $\bar\gamma$ by $\bar\tau_{ij} = \langle \bar E_i, \bar J \bar E_j \rangle$, $1 \leq i < j \leq d$. A helix of order d is called a *holomorphic helix of order* d if all the complex torsions are constant.

Using the Frenet equations, the bitension field of $\bar\gamma$ becomes

$$\begin{aligned} \tau_2(\bar\gamma) \quad = \quad & -3\bar\kappa_1\bar\kappa_1' \bar E_1 + (\bar\kappa_1'' - \bar\kappa_1^3 - \bar\kappa_1\bar\kappa_2^2 + \bar\kappa_1)\bar E_2 \\ & +(2\bar\kappa_1'\bar\kappa_2 + \bar\kappa_1\bar\kappa_2')\bar E_3 + \bar\kappa_1\bar\kappa_2\bar\kappa_3 \bar E_4 - 3\bar\kappa_1\bar\tau_{12}\bar J \bar E_1. \end{aligned} \tag{2.25}$$

In order to solve the biharmonic equation $\tau_2(\bar\gamma) = 0$, because of the last term in (2.25), we must split our study in three cases.

2.5.1 Biharmonic curves with $\bar\tau_{12} = \pm 1$

In this case $\bar J \bar E_2 = \pm E_1$ and, using the Frenet equations of $\bar\gamma$, we obtain

$$\bar J(\bar\nabla_{\bar E_1} \bar E_1) = \pm\bar\kappa_1 \bar E_1 = \bar\nabla_{\bar E_1}(\mp \bar E_2) = \mp \bar\nabla_{\bar E_1} \bar E_2,$$

so

$$\bar\nabla_{\bar E_1} \bar E_2 = -\bar\kappa_1 \bar E_1.$$

Consequently, $\bar\kappa_i = 0$, $i \geq 2$, and, from (2.25), we obtain the following.

Proposition 2.28 ([66]). *A Frenet curve $\bar\gamma : I \subset \mathbb{R} \to \mathbb{C}P^n$ parametrized by arc-length with $\bar\tau_{12} = \pm 1$ is proper-biharmonic if and only if it is a circle with $\bar\kappa_1 = 2$.*

Next, let us consider a curve $\bar\gamma : I \subset \mathbb{R} \to \mathbb{C}P^n$ parametrized by arc-length with $\bar\tau_{12} = \pm 1$, and denote by $\gamma : I \subset \mathbb{R} \to \mathbb{S}^{2n+1}$ one of its horizontal lifts. We shall characterize the biharmonicity of $\bar\gamma$ in terms of γ.

We denote by $\dot\nabla$ the Levi-Civita connection on $\mathbb{S}^{2n+1}$. We have $\gamma' = E_1 = (\bar E_1)^H$ and

$$\dot\nabla_{E_1} E_1 = (\bar\nabla_{\bar E_1} \bar E_1)^H = \bar\kappa_1 \bar E_2^H = \kappa_1 E_2,$$

i.e. $\kappa_1 = \bar{\kappa}_1$ and $E_2 = \bar{E}_2^H = \mp(\bar{J}\bar{E}_1)^H = \mp\hat{J}E_1$. It follows

$$\begin{aligned}
\dot{\nabla}_{E_1}E_2 &= (\bar{\nabla}_{\bar{E}_1}\bar{E}_2)^H + \langle\dot{\nabla}_{E_1}E_2,\xi\rangle\xi \\
&= -\kappa_1 E_1 - \langle E_2,\dot{\nabla}_{E_1}\xi\rangle\xi \\
&= -\kappa_1 E_1 \mp \langle E_2,E_2\rangle\xi \\
&= -\kappa_1 E_1 \mp \xi
\end{aligned}$$

and this means $\kappa_2 = 1$ and $E_3 = \mp\xi$. Then $\dot{\nabla}_{E_1}E_3 = \mp\dot{\nabla}_{E_1}\xi = -E_2$.
In conclusion γ is a helix with $\kappa_1 = \bar{\kappa}_1$ and $\kappa_2 = 1$.
Now, we have $\hat{J}\tau(\gamma) = \kappa_1\hat{J}E_2 = \pm\kappa_1 E_1$, which is tangent to γ, and then

$$\hat{J}\{(\hat{J}\tau(\gamma))^\top\} = \hat{J}^2\tau(\gamma) = -\tau(\gamma).$$

From

$$\begin{aligned}
\operatorname{div}\{(\bar{J}\tau(\bar{\gamma}))^\top\} &= \operatorname{div}\{\bar{\kappa}_1\langle\bar{J}\bar{E}_2,\bar{E}_1\rangle\bar{E}_1\} \\
&= \langle\bar{\nabla}_{\bar{E}_1}(\bar{\kappa}_1\langle\bar{J}\bar{E}_2,\bar{E}_1\rangle)\bar{E}_1,\bar{E}_1\rangle \\
&= \bar{\kappa}_1'\langle\bar{J}\bar{E}_2,\bar{E}_1\rangle + \bar{\kappa}_1\langle\bar{J}\bar{\nabla}_{\bar{E}_1}\bar{E}_2,\bar{E}_1\rangle \\
&= \pm\bar{\kappa}_1' = 0,
\end{aligned}$$

applying Remark 2.13 (iii), we have the following result.

Proposition 2.29 ([66]). *A Frenet curve $\bar{\gamma}: I \subset \mathbb{R} \to \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = \pm 1$ is proper-biharmonic if and only if its horizontal lift $\gamma : I \subset \mathbb{R} \to \mathbb{S}^{2n+1}$ is (-4)-biharmonic, i.e. γ is a helix with $\kappa_1 = 2$ and $\kappa_2 = 1$.*

Moreover, we can obtain the explicit parametric equations of the horizontal lifts of a proper-biharmonic Frenet curve $\bar{\gamma}: I \to \mathbb{C}P^n$.

Proposition 2.30 ([66]). *Let $\bar{\gamma}: I \subset \mathbb{R} \to \mathbb{C}P^n$ be a proper-biharmonic Frenet curve parametrized by arc-length with $\bar{\tau}_{12} = \pm 1$. Then its horizontal lift $\gamma : I \subset \mathbb{R} \to \mathbb{S}^{2n+1}$ can be parametrized in the Euclidean space $\mathbb{R}^{2n+2}$ by*

$$\begin{aligned}
\gamma(s) &= \frac{\sqrt{2-\sqrt{2}}}{2}\cos((\sqrt{2}+1)s)e_1 - \frac{\sqrt{2-\sqrt{2}}}{2}\sin((\sqrt{2}+1)s)\hat{J}e_1 \\
&\quad + \frac{\sqrt{2+\sqrt{2}}}{2}\cos((\sqrt{2}-1)s)e_3 + \frac{\sqrt{2+\sqrt{2}}}{2}\sin((\sqrt{2}-1)s)\hat{J}e_3,
\end{aligned}$$

where e_1 and e_3 are constant unit vectors in $\mathbb{R}^{2n+2}$ with e_3 orthogonal to e_1 and $\hat{J}e_1$.

Proof. The curve γ is a helix with the Frenet frame field $\{E_1 = \bar{E}_1^H, E_2 = \bar{E}_2^H, E_3 = \mp\xi\}$ and with curvatures $\kappa_1 = \bar{\kappa}_1 = 2$ and $\kappa_2 = 1$.

From the Weingarten equation of $\mathbb{S}^{2n+1}$ in $\mathbb{R}^{2n+2}$ and Frenet equations we get

$$\hat{\nabla}_{E_1}E_1 = \dot{\nabla}_{E_1}E_1 - \langle E_1,E_1\rangle\gamma = \kappa_1 E_2 - \gamma,$$

$$\hat{\nabla}_{E_1}\hat{\nabla}_{E_1}E_1 = \kappa_1\hat{\nabla}_{E_1}E_2 - E_1 = \kappa_1(-\kappa_1 E_1 \mp \xi) - E_1 = -(\kappa_1^2+1)E_1 \mp \kappa_1\xi$$

and

$$\begin{aligned}\hat{\nabla}_{E_1}\hat{\nabla}_{E_1}\hat{\nabla}_{E_1}E_1 &= -(\kappa_1^2+1)\hat{\nabla}_{E_1}E_1 \mp \kappa_1\hat{\nabla}_{E_1}\xi \\ &= -(\kappa_1^2+1)\hat{\nabla}_{E_1}E_1 - \kappa_1 E_2 \\ &= -6\gamma'' - \gamma.\end{aligned}$$

Hence γ is a solution of the differential equation

$$\gamma^{iv} + 6\gamma'' + \gamma = 0,$$

whose general solution is

$$\gamma(s) = \cos(As)c_1 + \sin(As)c_2 + \cos(Bs)c_3 + \sin(Bs)c_4,$$

where $A, B = \sqrt{2} \pm 1$ and $\{c_i\}$ are constant vectors in $\mathbb{R}^{2n+2}$.
As γ satisfies

$$\langle\gamma,\gamma\rangle = 1, \quad \langle\gamma',\gamma'\rangle = 1, \quad \langle\gamma,\gamma'\rangle = 0, \quad \langle\gamma',\gamma''\rangle = 0, \quad \langle\gamma'',\gamma''\rangle = 1+\kappa_1^2 = 5,$$

$$\langle\gamma,\gamma''\rangle = -1, \quad \langle\gamma',\gamma'''\rangle = -(1+\kappa_1^2) = -5, \quad \langle\gamma'',\gamma'''\rangle = 0,$$

$$\langle\gamma,\gamma'''\rangle = 0, \quad \langle\gamma''',\gamma'''\rangle = 7\kappa_1^2 + 1 = 29,$$

and since, in $s = 0$, we have $\gamma = c_1 + c_3$, $\gamma' = Ac_2 + Bc_4$, $\gamma'' = -A^2c_1 - B^2c_3$, $\gamma''' = -A^3c_2 - B^3c_4$, we obtain

$$c_{11} + 2c_{13} + c_{33} = 1 \tag{2.26}$$

$$A^2c_{22} + 2ABc_{24} + B^2c_{44} = 1 \tag{2.27}$$

$$Ac_{12} + Ac_{23} + Bc_{14} + Bc_{34} = 0 \tag{2.28}$$

$$A^3c_{12} + AB^2c_{23} + A^2Bc_{14} + B^3c_{34} = 0 \tag{2.29}$$

$$A^4c_{11} + 2A^2B^2c_{13} + B^4c_{33} = 5 \tag{2.30}$$

$$A^2c_{11} + (A^2+B^2)c_{13} + B^2c_{33} = 1 \tag{2.31}$$

$$A^4c_{22} + (AB^3 + A^3B)c_{24} + B^4c_{44} = 5 \tag{2.32}$$

$$A^5c_{12} + A^3B^2c_{23} + A^2B^3c_{14} + B^5c_{34} = 0 \tag{2.33}$$

$$A^3c_{12} + A^3c_{23} + B^3c_{14} + B^3c_{34} = 0 \tag{2.34}$$

$$A^6c_{22} + 2A^3B^3c_{24} + B^6c_{44} = 29 \tag{2.35}$$

where $c_{ij} = \langle c_i, c_j\rangle$. From (2.28), (2.29), (2.33) and (2.34) it follows that

$$c_{12} = c_{23} = c_{14} = c_{34} = 0.$$

The equations (2.26), (2.30) and (2.31) give

$$c_{11} = \frac{1-B^2}{A^2-B^2}, \quad c_{13} = 0, \quad c_{33} = \frac{A^2-1}{A^2-B^2}$$

and, from (2.27), (2.32) and (2.35), it follows that

$$c_{22} = \frac{1-B^2}{A^2-B^2}, \quad c_{24} = 0, \quad c_{44} = \frac{A^2-1}{A^2-B^2}.$$

Therefore, we obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{R}^{2n+2}$ with $|c_1| = |c_2| = \sqrt{\frac{1-B^2}{A^2-B^2}}$, $|c_3| = |c_4| = \sqrt{\frac{A^2-1}{A^2-B^2}}$.

By using that $E_1 = \gamma' \perp \xi$ and then that $\hat{J}E_2 = \pm E_1$, we conclude. □

Remark 2.31 ([66])**.** Under the flow-action of ξ, the (-4)-biharmonic curves γ induce the (-4)-biharmonic surfaces obtained in Example 2.20.

2.5.2 Biharmonic curves with $\bar{\tau}_{12} = 0$

From the expression (2.25) of the bitension field of $\bar{\gamma}$ we obtain that $\bar{\gamma}$ is proper-biharmonic if and only if

$$\begin{cases} \bar{\kappa}_1 = \text{constant} > 0, \quad \bar{\kappa}_2 = \text{constant} \\ \bar{\kappa}_1^2 + \bar{\kappa}_2^2 = 1 \\ \bar{\kappa}_2\bar{\kappa}_3 = 0 \end{cases}. \tag{2.36}$$

Proposition 2.32 ([66])**.** *A Frenet curve $\bar{\gamma} : I \subset \mathbb{R} \to \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = 0$ is proper-biharmonic if and only if either*

(a) $n = 2$ and $\bar{\gamma}$ is a circle with $\bar{\kappa}_1 = 1$,

or

(b) $n \geq 3$ and $\bar{\gamma}$ is a circle with $\bar{\kappa}_1 = 1$ or a helix with $\bar{\kappa}_1^2 + \bar{\kappa}_2^2 = 1$.

Proof. We only have to prove the statements concerning the dimension n.

First, since $\{\bar{E}_1, \bar{E}_2, \bar{J}\bar{E}_2\}$ are linearly independent, it follows that $n > 1$.

Now, assume that $\bar{\gamma}$ is a Frenet curve of osculating order 3 such that $\bar{J}\bar{E}_2 \perp \bar{E}_1$. We have

$$\begin{cases} \bar{E}_1 = \bar{\gamma}' \\ \bar{\nabla}_{\bar{E}_1}\bar{E}_1 = \bar{\kappa}_1\bar{E}_2 \\ \bar{\nabla}_{\bar{E}_1}\bar{E}_2 = -\bar{\kappa}_1\bar{E}_1 + \bar{\kappa}_2\bar{E}_3 \\ \bar{\nabla}_{\bar{E}_1}\bar{E}_3 = -\bar{\kappa}_2\bar{E}_2 \end{cases}. \tag{2.37}$$

It is easy to see that, at an arbitrary point, the system

$$S_1 = \{\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{J}\bar{E}_1, \bar{J}\bar{E}_2\}$$

consists of non-zero vectors which are orthogonal to each other, and therefore $n \geq 3$. □

Next, we shall consider the horizontal lift $\gamma : I \subset \mathbb{R} \to \mathbb{S}^{2n+1}$ of a curve $\bar{\gamma} : I \subset \mathbb{R} \to \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = 0$. As in the previous case we have $\gamma' = E_1 = \bar{E}_1^H$, $E_2 = \bar{E}_2^H$ and then $\hat{J}E_2 \perp E_1$. This means $\hat{J}(\tau(\gamma)) \perp E_1$, so $(\hat{J}(\tau(\gamma)))^\top = 0$. From Theorem 2.12 we obtain the following.

Proposition 2.33 ([66]). *A Frenet curve $\bar{\gamma}: I \subset \mathbb{R} \to \mathbb{C}P^n$ parametrized by arc-length with $\bar{\tau}_{12} = 0$ is proper-biharmonic if and only if its horizontal lift $\gamma: I \subset \mathbb{R} \to \mathbb{S}^{2n+1}$ is proper-biharmonic.*

The parametric equations of the proper-biharmonic Frenet curves in $\mathbb{S}^{2n+1}$ with $\hat{J}E_2 \perp E_1$ were obtained in [70]. Using that result we can state the following proposition.

Proposition 2.34 ([66]). *Let $\bar{\gamma}: I \subset \mathbb{R} \to \mathbb{C}P^n$ be a proper-biharmonic Frenet curve parametrized by arc-length with $\bar{\tau}_{12} = 0$. Then the horizontal lift $\gamma: I \subset \mathbb{R} \to \mathbb{S}^{2n+1}$ can be parametrized, in the Euclidean space $\mathbb{R}^{2n+2}$, either by*

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(\sqrt{2}s)e_1 + \frac{1}{\sqrt{2}}\sin(\sqrt{2}s)e_2 + \frac{1}{\sqrt{2}}e_3,$$

where $\{e_i, \hat{J}e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to each other, or by

$$\begin{aligned}\gamma(s) &= \tfrac{1}{\sqrt{2}}\cos(\sqrt{1+\kappa_1}s)e_1 + \tfrac{1}{\sqrt{2}}\sin(\sqrt{1+\kappa_1}s)e_2 \\ &\quad + \tfrac{1}{\sqrt{2}}\cos(\sqrt{1-\kappa_1}s)e_3 + \tfrac{1}{\sqrt{2}}\sin(\sqrt{1-\kappa_1}s)e_4,\end{aligned}$$

where $\kappa_1 \in (0,1)$, and $\{e_i, \hat{J}e_j\}_{i,j=1}^4$ are constant unit vectors orthogonal to each other.

2.5.3 Biharmonic curves with $\bar{\tau}_{12}$ different from 0, 1 or -1

Assume that $\bar{\gamma}$ is a proper-biharmonic Frenet curve of osculating order d such that $\bar{\tau}_{12}$ is different from $0, 1$ or -1.

First, we shall prove that $d \geq 4$.

Assume that $d = 2$. From the biharmonic equation $\tau_2(\bar{\gamma}) = 0$ we have $\bar{\kappa}_1 = \text{constant} > 0$ and then $(-\bar{\kappa}_1^3 + \bar{\kappa}_1)\bar{E}_2 - 3\bar{\kappa}_1\bar{\tau}_{12}\bar{J}\bar{E}_1 = 0$. It follows that $\bar{E}_2$ is parallel to $\bar{J}\bar{E}_1$, i.e. $\bar{\tau}_{12}^2 = 1$.

Now, if $d = 3$, from the biharmonic equation of $\bar{\gamma}$, we obtain again $\bar{\kappa}_1 = \text{constant} > 0$ and then

$$(-\bar{\kappa}_1^2 - \bar{\kappa}_2^2 + 1)\bar{E}_2 + \bar{\kappa}_2'\bar{E}_3 - 3\bar{\tau}_{12}\bar{J}\bar{E}_1 = 0. \tag{2.38}$$

Next, differentiating $-\bar{\tau}_{12}(s) = \langle \bar{E}_2, \bar{J}\bar{E}_1\rangle$, we obtain

$$\begin{aligned}-\bar{\tau}_{12}'(s) &= \langle \bar{\nabla}_{\bar{E}_1}\bar{E}_2, \bar{J}\bar{E}_1\rangle + \langle \bar{E}_2, \bar{\nabla}_{\bar{E}_1}\bar{J}\bar{E}_1\rangle = \langle \bar{\nabla}_{\bar{E}_1}\bar{E}_2, \bar{J}\bar{E}_1) + \langle \bar{E}_2, \bar{\kappa}_1\bar{J}\bar{E}_2) \\ &= \langle \bar{\nabla}_{\bar{E}_1}\bar{E}_2, \bar{J}\bar{E}_1\rangle = \langle -\bar{\kappa}_1\bar{E}_1 + \bar{\kappa}_2\bar{E}_3, \bar{J}\bar{E}_1\rangle \\ &= \bar{\kappa}_2\langle \bar{E}_3, \bar{J}\bar{E}_1\rangle.\end{aligned}$$

Hence, taking the inner product with $\bar{\kappa}_2\bar{E}_3$ in (2.38), we get $\bar{\kappa}_2'\bar{\kappa}_2 + 3\bar{\tau}_{12}\bar{\tau}_{12}' = 0$ and so $\bar{\kappa}_2^2 = -3\bar{\tau}_{12}^2 + \omega_0$, where $\omega_0 = \text{constant}$. Using (2.38) it results that $\bar{\kappa}_1^2 = 1 - \omega_0 + 6\bar{\tau}_{12}^2$. Therefore $f = \text{constant}$ and $\bar{\kappa}_2 = \text{constant}$. Finally, (2.38) becomes $(-\bar{\kappa}_1^2 - \bar{\kappa}_2^2 + 1)\bar{E}_2 - 3\bar{\tau}_{12}\bar{J}\bar{E}_1 = 0$, which means that $\bar{E}_2$ is parallel to $\bar{J}\bar{E}_1$.

We have proved the following result.

Proposition 2.35 ([66]). *Let $\bar{\gamma}$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^n$ of osculating order d, $1 \leq d \leq 2n$, with $\bar{\tau}_{12}$ different from 0, 1 or -1. Then we have $d \geq 4$.*

Next we shall prove that for a proper-biharmonic Frenet curve in $\mathbb{C}P^n$, $\bar{\tau}_{12}$ and $\bar{\kappa}_1$ are constants whatever the osculating order of $\bar{\gamma}$ is.
We have seen that $-\bar{\tau}_{12}'(s) = \bar{\kappa}_2\langle \bar{E}_3, J\bar{E}_1\rangle$. If $\tau_2(\bar{\gamma}) = 0$ we have $J\bar{E}_1 = \langle J\bar{E}_1, \bar{E}_2\rangle\bar{E}_2 + \langle J\bar{E}_1, \bar{E}_3\rangle\bar{E}_3 + \langle J\bar{E}_1, \bar{E}_4\rangle\bar{E}_4$ and

$$\begin{cases} \bar{\kappa}_1 = \text{constant} > 0 \\ \bar{\kappa}_1^2 + \bar{\kappa}_2^2 = 1 + 3\bar{\tau}_{12}^2 \\ \bar{\kappa}_2\bar{\kappa}_2' = -3\bar{\tau}_{12}\bar{\tau}_{12}' \\ \bar{\kappa}_2\bar{\kappa}_3 = 3\bar{\tau}_{12}\langle J\bar{E}_1, \bar{E}_4\rangle \end{cases}. \tag{2.39}$$

From the third equation of (2.39), we get

$$\bar{\kappa}_2^2 = -3\bar{\tau}_{12}^2 + \omega_0,$$

where $\omega_0 = $ constant. Replacing in the second equation of (2.39) it follows that

$$\bar{\kappa}_1^2 = 1 + 6\bar{\tau}_{12} - \omega_0,$$

which implies $\bar{\tau}_{12} = $ constant, and therefore, $\bar{\kappa}_2 = $ constant > 0. From $-\bar{\tau}_{12}'(s) = \bar{\kappa}_2\langle \bar{E}_3, J\bar{E}_1\rangle$, we have $\langle J\bar{E}_1, \bar{E}_3\rangle = 0$ and then $J\bar{E}_1 = f\bar{E}_2 + \langle J\bar{E}_1, \bar{E}_4\rangle\bar{E}_4$. It follows that there exists a unique constant $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $-\bar{\tau}_{12} = \cos\alpha_0$ and $\langle J\bar{E}_1, \bar{E}_4\rangle = \sin\alpha_0 = \frac{\bar{\kappa}_2\bar{\kappa}_3}{3\bar{\tau}_{12}}$.
We can summarise as follows.

Proposition 2.36 ([66]). *A Frenet curve $\bar{\gamma} : I \subset \mathbb{R} \to \mathbb{C}P^n$, $n \geq 2$, parametrized by arc-length with $\bar{\tau}_{12}$ different from 0, 1 or -1 is proper-biharmonic if and only if $J\bar{E}_1 = \cos\alpha_0\bar{E}_2 + \sin\alpha_0\bar{E}_4$ and*

$$\begin{cases} \bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3 = \text{constant} > 0 \\ \bar{\kappa}_1^2 + \bar{\kappa}_2^2 = 1 + 3\cos^2\alpha_0 \\ \bar{\kappa}_2\bar{\kappa}_3 = -\frac{3}{2}\sin(2\alpha_0) \\ \bar{\tau}_{12} = -\cos\alpha_0 \end{cases}, \tag{2.40}$$

where $\alpha_0 \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)$ is a constant.

We end this section classifying the proper-biharmonic curves in $\mathbb{C}P^n$ of osculating order $d \leq 4$. First, we prove the following proposition.

Proposition 2.37 ([66]). *Let $\bar{\gamma}$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^n$ of osculating order $d < 4$. Then $\bar{\gamma}$ is one of the following: a holomorphic circle of curvature $\bar{\kappa}_1 = 2$, a holomorphic circle of curvature $\bar{\kappa}_1 = 1$, or a holomorphic helix with $\bar{\kappa}_1^2 + \bar{\kappa}_2^2 = 1$.*

Proof. Let $\bar{\gamma}$ be a proper-biharmonic Frenet curve of osculating order $d < 4$. Then, from Proposition 2.35, $\bar{\tau}_{12} = \pm 1$ or $\bar{\tau}_{12} = 0$. If $\bar{\tau}_{12} = \pm 1$, from Proposition 2.28, $\bar{\gamma}$ is a circle of curvature $\bar{\kappa}_1 = 2$. If $\bar{\tau}_{12} = 0$ then we know that $\bar{\gamma}$ is either a holomorphic circle of curvature $\bar\kappa_1 = 1$ or a helix. We now prove that it is a holomorphic helix. For this we need to prove that the complex torsions $\bar\tau_{13}, \bar\tau_{23}$ are constant.

$$\begin{aligned}\bar\tau_{13} &= \langle \bar E_1, \bar J\bar E_3\rangle = -\frac{1}{\bar\kappa_2}\langle \bar\nabla_{\bar E_1}\bar E_2, \bar J\bar E_1\rangle = \frac{1}{\bar\kappa_2}\langle \bar E_2, \bar\nabla_{\bar E_1}\bar J\bar E_1\rangle\\ &= \frac{\bar\kappa_1}{\bar\kappa_2}\langle \bar E_2, \bar J\bar E_2\rangle = 0.\end{aligned}$$

Now, using that for a Frenet curve of osculating order 3 we have $\bar\kappa_1\bar\tau_{23} = \bar\tau_{13}' + \bar\kappa_2\bar\tau_{12}$, we see that also $\bar\tau_{23}$ is constant. □

When the biharmonic curve is of osculating order 4, system (2.40) has four solutions.

Proposition 2.38 ([66]). *Let $\bar\gamma$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^n$ of osculating order $d = 4$. Then $\bar\gamma$ is a holomorphic helix. Moreover, depending on the value of $\bar\tau_{12} = -\cos\alpha_0$, we have*

(a) If $\bar\tau_{12} > 0$, then the curvatures of $\bar\gamma$ are given by

$$\begin{cases}\bar\kappa_2 = \frac{\sin\alpha_0}{\sqrt{2}}\sqrt{1 - 3\cos^2\alpha_0 \pm \sqrt{9\cos^4\alpha_0 - 42\cos^2\alpha_0 + 1}}\\ \bar\kappa_3 = -\frac{3}{2\bar\kappa_2}\sin(2\alpha_0)\\ \bar\kappa_1 = -\frac{1}{\sin\alpha_0}(\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)\end{cases} \tag{2.41}$$

and

$$\bar\tau_{34} = -\bar\tau_{12} = \cos\alpha_0, \quad \bar\tau_{14} = -\bar\tau_{23} = -\sin\alpha_0 \quad \textit{and} \quad \bar\tau_{13} = \bar\tau_{24} = 0,$$

where $\alpha_0 \in (\frac{\pi}{2}, \arccos(-\frac{2-\sqrt{3}}{\sqrt{2}}))$.

(b) If $\bar\tau_{12} < 0$, then the curvatures of $\bar\gamma$ are given by

$$\begin{cases}\bar\kappa_2 = -\frac{\sin\alpha_0}{\sqrt{2}}\sqrt{1 - 3\cos^2\alpha_0 \pm \sqrt{9\cos^4\alpha_0 - 42\cos^2\alpha_0 + 1}}\\ \bar\kappa_3 = -\frac{3}{2\bar\kappa_2}\sin(2\alpha_0)\\ \bar\kappa_1 = -\frac{1}{\sin\alpha_0}(\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)\end{cases} \tag{2.42}$$

and

$$\bar\tau_{34} = -\bar\tau_{12} = \cos\alpha_0, \quad \bar\tau_{14} = -\bar\tau_{23} = -\sin\alpha_0 \quad \textit{and} \quad \bar\tau_{13} = \bar\tau_{24} = 0,$$

where $\alpha_0 \in (\frac{3\pi}{2}, \pi + \arccos(-\frac{2-\sqrt{3}}{\sqrt{2}}))$.

Proof. Let $\bar\gamma$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^n$ of osculating order $d = 4$. Then $\bar\tau_{12} = -\cos\alpha_0$ is different from 0, 1 or -1, and $\bar J\bar E_1 = \cos\alpha_0\bar E_2 + \sin\alpha_0\bar E_4$. Then it results that

$$\bar\tau_{12} = -\cos\alpha_0, \quad \bar\tau_{13} = 0, \quad \bar\tau_{14} = -\sin\alpha_0 \quad \text{and} \quad \bar\tau_{24} = 0.$$

In order to prove that $\bar\tau_{23}$ is constant we differentiate the expression of $\bar J\bar E_1$ and using the Frenet equations we obtain

$$\begin{aligned}\bar\nabla_{\bar E_1}\bar J\bar E_1 &= \cos\alpha_0\bar\nabla_{\bar E_1}\bar E_2 + \sin\alpha_0\bar\nabla_{\bar E_1}\bar E_4\\ &= -\bar\kappa_1\cos\alpha_0\bar E_1 + (\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)\bar E_3.\end{aligned}$$

On the other hand $\bar\nabla_{\bar E_1} J\bar E_1 = \bar\kappa_1 J\bar E_2$ and therefore we have

$$\bar\kappa_1 J\bar E_2 = -\bar\kappa_1 \cos\alpha_0 \bar E_1 + (\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)\bar E_3. \tag{2.43}$$

We take the inner product of (2.43) with $\bar E_3$, $J\bar E_2$ and $J\bar E_4$, respectively, and we get

$$\bar\kappa_1\bar\tau_{23} = -(\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0), \tag{2.44}$$

$$\bar\kappa_1\sin^2\alpha_0 = -(\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)\bar\tau_{23}, \tag{2.45}$$

$$0 = \bar\kappa_1\cos\alpha_0\sin\alpha_0 + (\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)\bar\tau_{34}. \tag{2.46}$$

From (2.44) and (2.45) we obtain

$$\bar\kappa_1^2\sin^2\alpha_0 = (\bar\kappa_2\cos\alpha_0 - \bar\kappa_3\sin\alpha_0)^2 \tag{2.47}$$

and $\tau_{23}^2 = \sin^2\alpha_0$. From $\tau_{23}^2 = \sin^2\alpha_0$, (2.44) and $\alpha_0 \in (\frac{\pi}{2},\pi)\cup(\frac{3\pi}{2},2\pi)$, one obtains

$$\bar\tau_{23} = \sin\alpha_0.$$

From $\bar\tau_{23} = \sin\alpha_0$, (2.44) and (2.46) we get

$$\bar\tau_{34} = \cos\alpha_0.$$

Finally, from Proposition 2.36 and (2.47) we obtain

$$\bar\kappa_2^4 + \bar\kappa_2^2\sin^2\alpha_0(3\cos^2\alpha_0 - 1) + 9\sin^4\alpha_0\cos^2\alpha_0 = 0.$$

The latter equation has either the solutions

$$\bar\kappa_2 = \frac{\sin\alpha_0}{\sqrt2}\sqrt{1 - 3\cos^2\alpha_0 \pm \sqrt{9\cos^4\alpha_0 - 42\cos^2\alpha_0 + 1}}$$

provided that $\alpha_0 \in (\frac{\pi}{2}, \arccos(-\frac{2-\sqrt3}{\sqrt2}))$, or the solutions

$$\bar\kappa_2 = -\frac{\sin\alpha_0}{\sqrt2}\sqrt{1 - 3\cos^2\alpha_0 \pm \sqrt{9\cos^4\alpha_0 - 42\cos^2\alpha_0 + 1}}$$

provided that $\alpha_0 \in (\frac{3\pi}{2}, \pi + \arccos(-\frac{2-\sqrt3}{\sqrt2}))$. Note that in both cases $\bar\kappa_2^2 \in (0,4)$, thus all solutions for $\bar\kappa_2$ are compatible with $\bar\kappa_1^2 + \bar\kappa_2^2 = 1 + 3\cos^2\alpha_0$. $\square$

Corollary 2.39 ([66])**.** *Any proper-biharmonic Frenet curve in $\mathbb{C}P^2$ is a holomorphic circle or a holomorphic helix of order* 4.

Remark 2.40 ([66])**.** The existence of biharmonic curves of osculating order $d \geq 4$ is an open problem (the case $d = 4$ and $n = 2$ will be solved in the next section). We note that there is no curve (not necessarily biharmonic) of order $d = 5$ in $\mathbb{C}P^n$ such that $J\bar E_1 = \cos\alpha_0\bar E_2 + \sin\alpha_0\bar E_4$, where $\alpha_0 \in (0,2\pi)\setminus\{\pi\}$.

2.6 Biharmonic curves in $\mathbb{C}P^2$

In this section we give the complete classification of all proper-biharmonic Frenet curves in $\mathbb{C}P^2$. From the previous section, we only have to classify the proper-biharmonic Frenet curves of osculating order 4.

In the proof of Proposition 2.38 we have seen that

$$\bar{\tau}_{34} = -\bar{\tau}_{12} = \cos\alpha_0, \quad \bar{\tau}_{14} = -\bar{\tau}_{23} = -\sin\alpha_0 \quad \text{and} \quad \bar{\tau}_{13} = \bar{\tau}_{24} = 0,$$

and

$$\bar{\kappa}_1 \sin\alpha_0 = -(\bar{\kappa}_2\cos\alpha_0 - \bar{\kappa}_3\sin\alpha_0),$$

which implies that $\bar{\kappa}_1 - \bar{\kappa}_3 = -\bar{\kappa}_2 \frac{\cos\alpha_0}{\sin\alpha_0} > 0$.
Moreover, if $\alpha_0 \in (\frac{\pi}{2}, \arccos(-\frac{2-\sqrt{3}}{\sqrt{2}}))$, then

$$\frac{\bar{\kappa}_1 - \bar{\kappa}_3}{\sqrt{\bar{\kappa}_2^2 + (\bar{\kappa}_1 - \bar{\kappa}_3)^2}} = -\cos\alpha_0 = \bar{\tau}_{12}, \quad \frac{\bar{\kappa}_2}{\sqrt{\bar{\kappa}_2^2 + (\bar{\kappa}_1 - \bar{\kappa}_3)^2}} = \sin\alpha_0 = \bar{\tau}_{23},$$

and, if $\alpha_0 \in (\frac{3\pi}{2}, \pi + \arccos(-\frac{2-\sqrt{3}}{\sqrt{2}}))$, then

$$\frac{\bar{\kappa}_1 - \bar{\kappa}_3}{\sqrt{\bar{\kappa}_2^2 + (\bar{\kappa}_1 - \bar{\kappa}_3)^2}} = \cos\alpha_0 = -\bar{\tau}_{12}, \quad \frac{\bar{\kappa}_2}{\sqrt{\bar{\kappa}_2^2 + (\bar{\kappa}_1 - \bar{\kappa}_3)^2}} = -\sin\alpha_0 = -\bar{\tau}_{23}.$$

In order to conclude, we briefly recall a result of S. Maeda and T. Adachi. In [91], they showed that for given positive constants $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$, there exist four equivalence classes of holomorphic helices of order 4 in $\mathbb{C}P^2$ with curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$ with respect to holomorphic isometries of $\mathbb{C}P^2$. The four classes are defined by certain relations on the complex torsions and they are: when $\bar{\kappa}_1 \neq \bar{\kappa}_3$

	$\bar{\kappa}_1 \neq \bar{\kappa}_3$		
I_1	$\bar{\tau}_{12} = \bar{\tau}_{34} = \mu$	$\bar{\tau}_{23} = \bar{\tau}_{14} = \bar{\kappa}_2\mu/(\bar{\kappa}_1 + \bar{\kappa}_3)$	$\bar{\tau}_{13} = \bar{\tau}_{24} = 0$
I_2	$\bar{\tau}_{12} = \bar{\tau}_{34} = -\mu$	$\bar{\tau}_{23} = \bar{\tau}_{14} = -\bar{\kappa}_2\mu/(\bar{\kappa}_1 + \bar{\kappa}_3)$	$\bar{\tau}_{13} = \bar{\tau}_{24} = 0$
I_3	$\bar{\tau}_{12} = -\bar{\tau}_{34} = \nu$	$\bar{\tau}_{23} = -\bar{\tau}_{14} = \bar{\kappa}_2\nu/(\bar{\kappa}_1 - \bar{\kappa}_3)$	$\bar{\tau}_{13} = \bar{\tau}_{24} = 0$
I_4	$\bar{\tau}_{12} = -\bar{\tau}_{34} = -\nu$	$\bar{\tau}_{23} = -\bar{\tau}_{14} = -\bar{\kappa}_2\nu/(\bar{\kappa}_1 - \bar{\kappa}_3)$	$\bar{\tau}_{13} = \bar{\tau}_{24} = 0$

where

$$\begin{cases} \mu = \dfrac{\bar{\kappa}_1 + \bar{\kappa}_3}{\sqrt{\bar{\kappa}_2^2 + (\bar{\kappa}_1 + \bar{\kappa}_3)^2}} \\ \nu = \dfrac{\bar{\kappa}_1 - \bar{\kappa}_3}{\sqrt{\bar{\kappa}_2^2 + (\bar{\kappa}_1 - \bar{\kappa}_3)^2}} \end{cases},$$

and when $\bar{\kappa}_1 = \bar{\kappa}_3$ the classes I_3 and I_4 are substituted by

	$\bar{\kappa}_1 = \bar{\kappa}_3$	
I'_3	$\bar{\tau}_{12} = \bar{\tau}_{34} = \bar{\tau}_{13} = \bar{\tau}_{24} = 0$	$\bar{\tau}_{23} = -\bar{\tau}_{14} = 1$
I'_4	$\bar{\tau}_{12} = \bar{\tau}_{34} = \bar{\tau}_{13} = \bar{\tau}_{24} = 0$	$\bar{\tau}_{23} = -\bar{\tau}_{14} = -1$

Using Maeda-Adachi classification, we can conclude.

Theorem 2.41 ([66]). *Let $\bar{\gamma}$ be a proper-biharmonic Frenet curve in $\mathbb{C}P^2$ of osculating order 4. Then $\bar{\gamma}$ is a holomorphic helix of order 4 of class I_3 or I_4 according to the following table*

I_3	if	$\bar{\tau}_{12} < 0$	and	$\bar{\tau}_{23} < 0$
I_4	if	$\bar{\tau}_{12} > 0$	and	$\bar{\tau}_{23} > 0$

Conversely,

(a) For any $\alpha_0 \in (\frac{\pi}{2}, \arccos(-\frac{2-\sqrt{3}}{\sqrt{2}}))$ there exist two proper-biharmonic holomorphic helices of order 4 of class I_3 with

$$\begin{cases} \bar{\kappa}_2 = \frac{\sin\alpha_0}{\sqrt{2}}\sqrt{1 - 3\cos^2\alpha_0 \pm \sqrt{9\cos^4\alpha_0 - 42\cos^2\alpha_0 + 1}} \\ \bar{\kappa}_3 = -\frac{3}{2\bar{\kappa}_2}\sin(2\alpha_0) \\ \bar{\kappa}_1 = -\frac{1}{\sin\alpha_0}(\bar{\kappa}_2\cos\alpha_0 - \bar{\kappa}_3\sin\alpha_0) \end{cases}. \tag{2.48}$$

(b) For any $\alpha_0 \in (\frac{3\pi}{2}, \pi + \arccos(-\frac{2-\sqrt{3}}{\sqrt{2}}))$ there exist two proper-biharmonic holomorphic helices of order 4 of class I_4 with

$$\begin{cases} \bar{\kappa}_2 = -\frac{\sin\alpha_0}{\sqrt{2}}\sqrt{1 - 3\cos^2\alpha_0 \pm \sqrt{9\cos^4\alpha_0 - 42\cos^2\alpha_0 + 1}} \\ \bar{\kappa}_3 = -\frac{3}{2\bar{\kappa}_2}\sin(2\alpha_0) \\ \bar{\kappa}_1 = -\frac{1}{\sin\alpha_0}(\bar{\kappa}_2\cos\alpha_0 - \bar{\kappa}_3\sin\alpha_0) \end{cases}. \tag{2.49}$$

Chapter 3

Biharmonic submanifolds in Sasakian space forms

The present chapter, divided in three sections, is dedicated to the study of biharmonic submanifolds in Sasakian space forms.

3.1 Explicit formulas for biharmonic submanifolds in Sasakian space forms

3.1.1 Introduction

At the beginning of the first section we classify all proper-biharmonic Legendre curves in any dimensional Sasakian space forms. Because of the complexity of the biharmonic equation, we had to do a case by case analysis and the classification is given by Theorems 3.4, 3.7, 3.8 and 3.10. As a by-product we prove that in a 5-dimensional Sasakian space form all proper-biharmonic curves are helices (Theorem 3.13). Then, we consider the unit $(2n+1)$-dimensional Euclidian sphere $\mathbb{S}^{2n+1}$ endowed with the canonical and deformed Sasakian structures defined by S. Tanno as a model for the Sasakian space forms, and obtain the explicit parametric equations of proper-biharmonic Legendre curves (Theorems 3.17, 3.19 and 3.20).

In the second part of the first section we prove that, by composing with the flow of the characteristic vector field of a Sasakian space form, we can render a proper-biharmonic integral submanifold onto a proper-biharmonic anti-invariant submanifold (Theorem 3.22). This result allows us to obtain all proper-biharmonic surfaces which are invariant under the flow-action of the characteristic vector field (Theorem 3.24).

3.1.2 Preliminaries

In this section we briefly recall basic things from the theory of Sasakian manifolds (for example see [26]) which we shall use throughout the paper.
A *contact metric structure* on an odd-dimensional manifold N^{2n+1} is given by (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$ on N, ξ is a vector field, η is an 1-form and g is

a Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X, Y), \quad \forall X, Y \in C(TN).$$

A contact metric manifold $(N, \varphi, \xi, \eta, g)$ is called *Sasakian* if it is *normal*, i.e.

$$N_\varphi + 2d\eta \otimes \xi = 0,$$

where

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], \quad \forall X, Y \in C(TN),$$

is the Nijenhuis tensor field of φ, or, equivalently, if

$$(\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad \forall X, Y \in C(TN).$$

We note that from the above formula it follows $\nabla_X \xi = -\varphi X$.

The *contact distribution* of a Sasakian manifold $(N, \varphi, \xi, \eta, g)$ is defined by $\{X \in TN : \eta(X) = 0\}$. We say that a submanifold M of N is an *integral submanifold* if $\eta(X) = 0$ for any vector X tangent to M; in particular, an integral curve is called a *Legendre* curve. The maximum dimension for an integral submanifold of N^{2n+1} is n. Moreover, for $m = n$, one gets $\varphi(NM) = TM$. If we denote by B the second fundamental form of M then, by a straightforward computation, one obtains the following relation which we shall use later in this chapter

$$g(B(X, Y), \varphi Z) = g(B(X, Z), \varphi Y),$$

for any vector fields X, Y and Z tangent to M (see also [10], [130]).

A submanifold $\widetilde{M}$ of N which is tangent to ξ is said to be *anti-invariant* if φ maps any vector tangent to $\widetilde{M}$ and normal to ξ to a vector normal to $\widetilde{M}$.

Let $(N, \varphi, \xi, \eta, g)$ be a Sasakian manifold. The sectional curvature of a 2-plane generated by X and φX, where X is an unit vector orthogonal to ξ, is called the *φ-sectional curvature* determined by X. If the φ-sectional curvature is a constant c, then $(N, \varphi, \xi, \eta, g)$ is called a *Sasakian space form* and it is denoted by $N(c)$.
The curvature tensor field of a Sasakian space form $N(c)$ is given by

$$\begin{aligned} R(X, Y)Z = \ & \tfrac{c+3}{4}\{g(Z, Y)X - g(Z, X)Y\} + \tfrac{c-1}{4}\{\eta(Z)\eta(X)Y - \\ & -\eta(Z)\eta(Y)X + g(Z, X)\eta(Y)\xi - g(Z, Y)\eta(X)\xi + \\ & +g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\}. \end{aligned} \tag{3.1}$$

The classification of complete, simply connected Sasakian space forms $N(c)$ was given in [125]. When $c > -3$, $N(c)$ is isometric to the unit sphere $\mathbb{S}^{2n+1}$ endowed with the Sasakian structure defined by S. Tanno. This structure is given as follows (see [126]).

Let $\mathbb{S}^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$ be the unit $(2n+1)$-dimensional Euclidean sphere. Consider the following structure tensor fields on $\mathbb{S}^{2n+1}$: the standard metric field g_0, the vector field $\xi_0(z) = -\mathcal{J}z$, $z \in \mathbb{S}^{2n+1}$, where $\mathcal{J}$ is the usual almost complex structure on $\mathbb{C}^{n+1}$ defined by

$$\mathcal{J}z = (-y^1, ..., -y^{n+1}, x^1, ..., x^{n+1}),$$

for $z = (x^1, ..., x^{n+1}, y^1, ..., y^{n+1})$, and $\varphi_0 = s \circ \mathcal{J}$, where $s : T_z\mathbb{C}^{n+1} \to T_z\mathbb{S}^{2n+1}$ denotes the orthogonal projection. Equipped with these tensors, $\mathbb{S}^{2n+1}$ becomes a Sasakian space form with the φ_0-sectional curvature equal to 1, denoted by $\mathbb{S}^{2n+1}(1)$.
Now, consider the deformed structure on $\mathbb{S}^{2n+1}$

$$\eta = a\eta_0, \quad \xi = \frac{1}{a}\xi_0, \quad \varphi = \varphi_0, \quad g = ag_0 + a(a-1)\eta_0 \otimes \eta_0,$$

where a is a positive constant. The structure (φ, ξ, η, g) is still a Sasakian structure and $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form with constant φ-sectional curvature $c = \frac{4}{a} - 3$, $c > -3$, denoted by $\mathbb{S}^{2n+1}(c)$.

If M^m, with $m \leq n$, is a submanifold of the sphere $\mathbb{S}^{2n+1}$ then M is integral with respect to its canonical Sasakian structure $(\varphi_0, \xi_0, \eta_0, g_0)$ if and only if it is integral with respect to the deformed one (φ, ξ, η, g), since $\eta_0(X) = 0$ if and only if $\eta(X) = 0$ for any vector field X tangent to M. Moreover, if M is an integral submanifold of $\mathbb{S}^{2n+1}$ then the normal bundle of M in $(\mathbb{S}^{2n+1}, g_0)$ coincides with the normal bundle of M in $(\mathbb{S}^{2n+1}, g)$, since for any $X \in T_pM$ and $Y \in T_p\mathbb{S}^{2n+1}$, where p is an arbitrary point in M, we have $g_0(X, Y) = 0$ if and only if $g(X, Y) = 0$.

Next, we consider M to be an integral submanifold of $\mathbb{S}^{2n+1}$, and denote by g_0^M and g^M the induced metrics on M by g_0 and g, respectively. Denote by $\dot{\nabla}^M$ and ∇^M their Levi-Civita connections. Then the identity map $\mathbf{1} : (M, g_0^M) \to (M, g^M)$ is a homothety and therefore $\dot{\nabla}^M = \nabla^M$.

The following Lemma holds.

Lemma 3.1. *Let M be an integral submanifold of $\mathbb{S}^{2n+1}$. If X and Y are vector fields tangent to M then*

$$\dot{\nabla}_X Y = \nabla_X Y \quad \textit{and} \quad \dot{\nabla}_X \varphi Y = \nabla_X \varphi Y,$$

where $\dot{\nabla}$ and ∇ are the Levi-Civita connections on $(\mathbb{S}^{2n+1}, g_0)$ and $(\mathbb{S}^{2n+1}, g)$, respectively.

Proof. From the definition of the metric g we have, for any vector fields X, Y tangent to M and Z tangent to $\mathbb{S}^{2n+1}$,

$$g(\nabla_X Y, Z) = ag_0(\nabla_X Y, Z) + a(a-1)\eta_0(\nabla_X Y)\eta_0(Z).$$

But, since M is integral,

$$\eta_0(\nabla_X Y) = \frac{1}{a}\eta(\nabla_X Y) = \frac{1}{a}g(\nabla_X Y, \xi) = -\frac{1}{a}g(Y, \nabla_X \xi) = \frac{1}{a}g(Y, \varphi X) = 0,$$

and so

$$g(\nabla_X Y, Z) = ag_0(\nabla_X Y, Z).$$

On the other hand, applying the characterization of the Levi-Civita connection for ∇ and $\dot{\nabla}$, we obtain

$$g(\nabla_X Y, Z) = a g_0(\dot{\nabla}_X Y, Z).$$

From the last two relations we get

$$g_0(\nabla_X Y, Z) = g_0(\dot{\nabla}_X Y, Z)$$

and therefore $\dot{\nabla}_X Y = \nabla_X Y$ for any vector fields X and Y tangent to M.

For the second relation, we use $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$ and $(\dot{\nabla}_X \varphi)Y = g_0(X,Y)\xi_0 - \eta_0(Y)X$ for vector fields X and Y tangent to M, and come to the conclusion. □

We end this subsection recalling that a contact metric manifold $(N, \varphi, \xi, \eta, g)$ is *regular* if for any point $p \in N$ there exists a cubic neighborhood such that any integral curve of ξ passes through it at most once; and it is *strictly regular* if all integral curves of ξ are homeomorphic to each other.
Let $(N, \varphi, \xi, \eta, g)$ be a regular contact metric manifold. Then the orbit space $\bar{N} = N/\xi$ has a natural manifold structure and, moreover, if N is compact then N is a principal circle bundle over $\bar{N}$ (the Boothby-Wang Theorem). In this case the fibration $\pi : N \to \bar{N}$ is called *the Boothby-Wang fibration.* The Hopf fibration $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{C}P^n(4)$ is a well-known example of a Boothby-Wang fibration.

Theorem 3.2 ([106])**.** *Let $(N, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian manifold. Then on $\bar{N}$ can be given the structure of a Kähler manifold. Moreover, if $(N, \varphi, \xi, \eta, g)$ is a Sasakian space form $N(c)$, then $\bar{N}$ has constant sectional holomorphic curvature $c + 3$.*

Even if N is non-compact, we still call the fibration $\pi : N \to \bar{N}$ of a strictly regular Sasakian manifold, the Boothby-Wang fibration.

We end with the following classification result.

Theorem 3.3 ([135])**.** *A simply connected complete Kähler manifold of constant holomorphic sectional curvature c can be identified with the complex projective space $\mathbb{C}P^n$, the open unit ball D^n in $\mathbb{C}^n$, or $\mathbb{C}^n$, according as $c > 0$, $c < 0$, or $c = 0$.*

3.1.3 Biharmonic Legendre curves in Sasakian space forms

We shall work with Frenet curves of osculating order r, parametrized by arc-length. For such a curve $\gamma : I \to N$ we shall denote by $\{E_1 = \gamma' = T, E_2, \dots, E_r\}$ the Frenet frame field along it, and by $\kappa_1, \dots, \kappa_{r-1}$ the corresponding curvatures which are positive functions on I.

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. Since

$$\begin{aligned}\nabla_T^3 T = \ & (-3\kappa_1\kappa_1')E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 \\ & + \kappa_1\kappa_2\kappa_3 E_4\end{aligned}$$

and

$$R(T, \nabla_T T)T = -\frac{(c+3)\kappa_1}{4}E_2 - \frac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T,$$

we obtain the expression of the bitension vector field

$$\begin{aligned}\tau_2(\gamma) = \quad & \nabla_T^3 T - R(T, \nabla_T T)T \\ = \quad & (-3\kappa_1\kappa_1')E_1 + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \tfrac{(c+3)\kappa_1}{4}\right)E_2 \\ & +(2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 + \tfrac{3(c-1)\kappa_1}{4} g(E_2, \varphi T)\varphi T.\end{aligned} \tag{3.2}$$

In the following we shall solve the biharmonic equation $\tau_2(\gamma) = 0$. Because of the last term of $\tau_2(\gamma)$ we must do a case by case analysis.

Case I: $\mathbf{c = 1}$.

In this case, from (3.2), it follows that γ is proper-biharmonic if and only if

$$\begin{cases} \kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant} \\ \kappa_1^2 + \kappa_2^2 = 1 \\ \kappa_2\kappa_3 = 0 \end{cases}.$$

One obtains the following.

Theorem 3.4 ([70]). *Let $N^{2n+1}(1)$ be a Sasakian space form and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r. Then γ is proper-biharmonic if and only if $n \geq 2$ and either γ is a circle with $\kappa_1 = 1$, or γ a helix with $\kappa_1^2 + \kappa_2^2 = 1$.*

Remark 3.5 ([70]). If $n = 1$ and γ is a non-geodesic Legendre curve we have $\nabla_T T = \pm\kappa_1\varphi T$ and then $E_2 = \pm\varphi T$ and $\nabla_T E_2 = \pm\nabla_T\varphi T = \pm(\xi\mp\kappa_1 T) = -\kappa_1 T\pm\xi$. Therefore $\kappa_2 = 1$ and γ cannot be biharmonic.

Case II: $\mathbf{c \neq 1}$, $\mathbf{E_2 \perp \varphi T}$.

From (3.2) we obtain that γ is proper-biharmonic if and only if

$$\begin{cases} \kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant} \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} \\ \kappa_2\kappa_3 = 0 \end{cases}.$$

Before stating the theorem we need the following lemma which imposes a restriction on the dimension of the manifold $N^{2n+1}(c)$.

Lemma 3.6 ([70]). *Let γ be a Legendre Frenet curve of osculating order 3 such that $E_2 \perp \varphi T$. Then $\{T = E_1, E_2, E_3, \varphi T, \xi, \nabla_T\varphi T\}$ is linearly independent, in any point, and hence $n \geq 3$.*

Proof. Since γ is a Frenet curve of osculating order 3, we have

$$\begin{cases} E_1 = \gamma' = T \\ \nabla_T E_1 = \kappa_1 E_2 \\ \nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3 \\ \nabla_T E_3 = -\kappa_2 E_2 \end{cases}.$$

It is easy to see that, in an arbitrary point, the system

$$S_1 = \{T = E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T\}$$

has only non-zero vectors and

$$T \perp E_2, \quad T \perp E_3, \quad T \perp \varphi T, \quad T \perp \xi, \quad T \perp \nabla_T \varphi T.$$

Thus S_1 is linearly independent if and only if $S_2 = \{E_2, E_3, \varphi T, \xi, \nabla_T \varphi T\}$ is linearly independent. Further, as

$$E_2 \perp \xi,\ E_2 \perp \nabla_T \varphi T,\ E_3 \perp \xi,\ E_3 \perp \nabla_T \varphi T,\ \varphi T \perp \xi,\ \varphi T \perp \nabla_T \varphi T,$$

and

$$E_2 \perp E_3 \perp \varphi T,$$

it follows that S_2 is linearly independent if and only if $S_3 = \{\xi, \nabla_T \varphi T\}$ is linearly independent. But $\nabla_T \varphi T = \xi + \kappa_1 \varphi E_2$, $\kappa_1 \neq 0$, and therefore S_3 is linearly independent. □

Now we can state the result.

Theorem 3.7 ([70]). *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r such that $E_2 \perp \varphi T$. We have*
1) If $c \leq -3$ then γ is biharmonic if and only if it is a geodesic.
2) If $c > -3$ then γ is proper-biharmonic if and only if either
a) $n \geq 2$ and γ is a circle with $\kappa_1^2 = \frac{c+3}{4}$. In this case $\{E_1, E_2, \varphi T, \xi\}$ are linearly independent,
or
b) $n \geq 3$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4}$. In this case $\{E_1, E_2, E_3, \varphi T, \xi, \nabla_T \varphi T\}$ are linearly independent.

Case III: $\mathbf{c \neq 1}$, $\mathbf{E_2 \parallel \varphi T}$.
In this case, from (3.2), γ is proper-biharmonic if and only if

$$\begin{cases} \kappa_1 = \text{constant} > 0, \quad \kappa_2 = \text{constant} \\ \\ \kappa_1^2 + \kappa_2^2 = c \\ \\ \kappa_2 \kappa_3 = 0 \end{cases}.$$

We can assume that $E_2 = \varphi T$. Then we have $\nabla_T T = \kappa_1 E_2 = \kappa_1 \varphi T$, $\nabla_T E_2 = \nabla_T \varphi T = \xi - \kappa_1 T$. That means $E_3 = \xi$ and $\kappa_2 = 1$. Hence $\nabla_T E_3 = \nabla_T \xi = -\varphi T = -E_2$. Therefore, we obtain the following result.

Theorem 3.8 ([70]). *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$ and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r such that $E_2 \parallel \varphi T$. Then $\{T, \varphi T, \xi\}$ is the Frenet frame field of γ and we have*
1) If $c < 1$ then γ is biharmonic if and only if it is a geodesic.
2) If $c > 1$ then γ is proper-biharmonic if and only if it is a helix with $\kappa_1^2 = c - 1$ (and $\kappa_2 = 1$).

Remark 3.9. If $n=1$, for any Legendre curve $E_2 \parallel \varphi T$, and we reobtain Inoguchi's result in [79].

Case IV: $\mathbf{c} \neq \mathbf{1}$ and $\mathbf{g}(\mathbf{E_2}, \varphi\mathbf{T})$ is not constant $\mathbf{0}, \mathbf{1}$ or $-\mathbf{1}$.
Assume that γ is a proper-biharmonic Legendre Frenet curve of osculating order r such that $g(E_2, \varphi T)$ is not constant $0, 1$ or -1. One can check that, in this case, $4 \leq r \leq 2n+1$, $n \geq 2$, and $\varphi T \in \operatorname{span}\{E_2, E_3, E_4\}$.

Now, we denote $f(s) = g(E_2, \varphi T)$ and differentiating it we obtain

$$\begin{aligned} f'(s) &= g(\nabla_T E_2, \varphi T) + g(E_2, \nabla_T \varphi T) = g(\nabla_T E_2, \varphi T) + g(E_2, \xi + \kappa_1 \varphi E_2) \\ &= g(\nabla_T E_2, \varphi T) = g(-\kappa_1 T + \kappa_2 E_3, \varphi T) \\ &= \kappa_2 g(E_3, \varphi T). \end{aligned}$$

Since $\varphi T = g(\varphi T, E_2)E_2 + g(\varphi T, E_3)E_3 + g(\varphi T, E_4)E_4$, the curve γ is proper-biharmonic if and only if

$$\begin{cases} \kappa_1 = \text{constant} > 0 \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4} f^2 \\ \kappa_2' = -\frac{3(c-1)}{4} f g(\varphi T, E_3) \\ \kappa_2 \kappa_3 = -\frac{3(c-1)}{4} f g(\varphi T, E_4) \end{cases}.$$

Using the expression of $f'(s)$ we see that the third equation of the above system is equivalent to

$$\kappa_2^2 = -\frac{3(c-1)}{4} f^2 + \omega_0,$$

where $\omega_0 = $ constant. Replacing in the second equation it follows

$$\kappa_1^2 = \frac{c+3}{4} - \omega_0 + \frac{3(c-1)}{2} f^2,$$

which implies $f = $ constant. Thus $\kappa_2 = $ constant > 0, $g(E_3, \varphi T) = 0$ and then $\varphi T = f E_2 + g(\varphi T, E_4) E_4$. It follows that there exists an unique constant $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ such that $f = \cos\alpha_0$ and $g(\varphi T, E_4) = \sin\alpha_0$.
We can state the result.

Theorem 3.10 ([70]). *Let $N^{2n+1}(c)$ be a Sasakian space form with $c \neq 1$, $n \geq 2$, and $\gamma : I \to N$ a Legendre Frenet curve of osculating order r such that $g(E_2, \varphi T)$ is not constant $0, 1$ or -1. We have*

1) If $c \leq -3$ then γ is biharmonic if and only if it is a geodesic.

2) If $c > -3$ then γ is proper-biharmonic if and only if $r \geq 4$, $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$ and

$$\begin{cases} \kappa_1, \kappa_2, \kappa_3 = \text{constant} > 0 \\ \kappa_1^2 + \kappa_2^2 = \frac{c+3}{4} + \frac{3(c-1)}{4}\cos^2\alpha_0 \\ \kappa_2\kappa_3 = -\frac{3(c-1)}{8}\sin(2\alpha_0) \end{cases},$$

where $\alpha_0 \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ *is a constant such that* $c + 3 + 3(c-1)\cos^2 \alpha_0 > 0$ *and* $3(c-1)\sin(2\alpha_0) < 0$.

Remark 3.11 ([70]). In this case we may obtain biharmonic curves which are not helices.

Proposition 3.12 ([70]). *Assume that* $c > -3$, $c \neq 1$, *and* $n = 2$. *Let* γ *be a proper-biharmonic Legendre Frenet curve of osculating order* r, *such that* $g(E_2, \varphi T)$ *is not constant 0,-1, or 1. Then* γ *is a helix of order* 4 *or* 5.

Proof. We know that $r \in \{4, 5\}$. If $r = 4$, then the result is obvious from Theorem 3.10. Assume now $r = 5$. Since $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$, and $\xi \perp \varphi T$, $\xi \perp E_2$, we get $\xi \perp E_4$, and then, along γ, $\xi \in \operatorname{span}\{E_3, E_5\}$.
From the Frenet equations of γ it follows that

$$g(\nabla_T E_3, \xi) = g(-\kappa_2 E_2 + \kappa_3 E_4, \xi) = 0$$

and

$$g(\nabla_T E_5, \xi) = g(-\kappa_4 E_4, \xi) = 0.$$

Then, since $\nabla g = 0$, we obtain $(g(E_3, \xi))' = 0$ and $(g(E_5, \xi))' = 0$, i.e. $a = g(E_3, \xi) =$ constant and $b = g(E_5, \xi) =$ constant.
Now, we have

$$g(\nabla_T E_4, \xi) = -\kappa_3 g(E_3, \xi) + \kappa_4 g(E_5, \xi) = -\kappa_3 a + \kappa_4 b$$

and, since $g(\nabla_T E_4, \xi) = g(E_4, \varphi T) = \sin\alpha_0$, we get

$$\sin\alpha_0 = -\kappa_3 a + \kappa_4 b \tag{3.3}$$

which implies that $b = 0$ or $\kappa_4 =$ constant.
Case $b = 0$. Since $\xi \in \operatorname{span}\{E_3, E_5\}$, we have $E_3 = \pm\xi$ and therefore

$$\nabla_T E_3 = \mp\varphi T = \mp\cos\alpha_0 E_2 \mp \sin\alpha_0 E_4.$$

From the third Frenet equation, $\kappa_2 = \pm\cos\alpha_0$, $\kappa_3 = \mp\sin\alpha_0$, and then, from Theorem 3.10, $\kappa_2\kappa_3 = -\frac{1}{2}\sin(2\alpha_0) = -\frac{3(c-1)}{8}\sin(2\alpha_0)$. Thus, we have $c = \frac{7}{3}$ and, again using Theorem 3.10, $\kappa_1 = \frac{2}{\sqrt{3}}$.

We shall prove now $\kappa_4 = \kappa_1$, so γ is a helix of order 5. From the last Frenet equation, we obtain

$$g(\nabla_T E_5, \varphi T) = -\kappa_4 g(E_4, \varphi T) = -\kappa_4 \sin\alpha_0. \tag{3.4}$$

Since $g(E_5, \varphi T) = 0$ we have $g(\nabla_T E_5, \varphi T) + g(E_5, \nabla_T \varphi T) = 0$. We can check that $g(E_5, \nabla_T \varphi T) = \kappa_1 g(E_5, \varphi E_2)$, therefore, using (3.4), we get

$$\kappa_1 g(E_5, \varphi E_2) = \kappa_4 \sin\alpha_0. \tag{3.5}$$

Next, from the fourth Frenet equation and (3.5),

$$g(\nabla_T E_4, \varphi E_2) = \kappa_4 g(E_5, \varphi E_2) = \frac{\kappa_4^2}{\kappa_1}\sin\alpha_0. \tag{3.6}$$

Since $\varphi T = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$ it results that $g(E_4, \varphi E_2) = 0$. It follows

$$\begin{aligned} g(\nabla_T E_4, \varphi E_2) &= -g(E_4, \nabla_T \varphi E_2) \\ &= -g(E_4, \varphi \nabla_T E_2) = \kappa_1 g(E_4, \varphi T) \\ &= \kappa_1 \sin\alpha_0. \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) we obtain $\kappa_4 = \kappa_1 = \frac{2}{\sqrt{3}}$.
Case $b \neq 0$. Of course, due to (3.3) $\kappa_4 =$ constant and so γ is a helix. Moreover, we can obtain an additional relation between the curvatures.
Indeed, since $\xi \in \operatorname{span}\{E_3, E_5\}$ it follows $a^2 + b^2 = 1$. On the other hand

$$\begin{aligned} g(\nabla_T E_2, \xi) &= g(E_2, \varphi T) = \cos\alpha_0 \\ &= g(-\kappa_1 T + \kappa_2 E_3, \xi) = \kappa_2 a \end{aligned}$$

and as $-\kappa_3 a + \kappa_4 b = \sin\alpha_0$, replacing in $a^2 + b^2 = 1$ we get

$$(\kappa_2 \sin\alpha_0 + \kappa_3 \cos\alpha_0)^2 + \kappa_4^2 (\cos\alpha_0)^2 = \kappa_2^2 \kappa_4^2.$$

□

From Theorems 3.4, 3.7, 3.8 and the above Proposition we conclude.

Theorem 3.13 ([70]). *Let γ be a proper-biharmonic Legendre curve in $N^5(c)$. Then $c > -3$ and γ is a helix of order r with $2 \leq r \leq 5$.*

In the following, we shall choose the unit $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1}$ with its canonical and deformed Sasakian structures as a model for the complete, simply connected Sasakian space form with constant φ-sectional curvature $c > -3$, and we shall find the explicit equations of biharmonic Legendre curves obtained in the first three cases, viewed as curves in $\mathbb{R}^{2n+2}$.

In [92] are introduced the complex torsions for a Frenet curve in a complex manifold. In the same way, for $\gamma : I \to N$ a Legendre Frenet curve of osculating order r in a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$, we define the φ-torsions $\tau_{ij} = g(E_i, \varphi E_j) = -g(\varphi E_i, E_j)$, $i, j = 1, \ldots, r$, $i < j$.

It is easy to see that the following holds.

Proposition 3.14. *Let $\gamma : I \to N(c)$ be a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$, $c \neq 1$. Then $c > -3$ and τ_{12} is constant.*

Moreover, we prove the following result.

Proposition 3.15. *If γ is a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$, $c > -3$, $c \neq 1$, of osculating order $r < 4$, then it is a circle or a helix with constant φ-torsions.*

Proof. From Theorems 3.7, 3.8 and 3.10 we see that if γ is a proper-biharmonic Legendre Frenet curve of osculating order $r < 4$, then $\tau_{12} = 0$ or $\tau_{12} = \pm 1$ and, obviously, we only have to prove that when γ is a helix then τ_{13} and τ_{23} are constants.

Indeed, by using the Frenet equations of γ, we have

$$\begin{aligned}\tau_{13} &= g(E_1, \varphi E_3) = -\frac{1}{\kappa_2} g(\varphi E_1, \nabla_{E_1} E_2 + \kappa_1 E_1) = -\frac{1}{\kappa_2} g(\varphi E_1, \nabla_{E_1} E_2) \\ &= \frac{1}{\kappa_2} g(E_2, \nabla_{E_1} \varphi E_1) = \frac{1}{\kappa_2} g(E_2, \varphi \nabla_{E_1} E_1 + \xi) = 0\end{aligned}$$

since

$$g(E_2, \xi) = \frac{1}{\kappa_1} g(\nabla_{E_1} E_1, \xi) = -\frac{1}{\kappa_1} g(E_1, \nabla_{E_1} \xi) = \frac{1}{\kappa_1} g(E_1, \varphi E_1) = 0.$$

On the other hand, it is easy to see that for any Frenet curve of osculating order 3 we have $\tau_{23} = \frac{1}{\kappa_1}(\tau_{13}' + \kappa_2 \tau_{12} + \eta(E_3))$ and

$$\begin{aligned}\eta(E_3) &= g(E_3, \xi) = \frac{1}{\kappa_2}\Big(g(\nabla_{E_1} E_2, \xi) + \kappa_1 g(E_1, \xi)\Big) = -\frac{1}{\kappa_2} g(E_2, \nabla_{E_1} \xi) \\ &= -\frac{1}{\kappa_2} \tau_{12}.\end{aligned}$$

In conclusion, $\tau_{23} = \frac{1}{\kappa_1}(\tau_{13}' + \kappa_2 \tau_{12} - \frac{1}{\kappa_2} \tau_{12}) = \text{constant}$.

□

Proposition 3.16. *If γ is a proper-biharmonic Legendre Frenet curve in a Sasakian space form $N(c)$ of osculating order $r = 4$, then $c \in (\frac{7}{3}, 5)$ and the curvatures of γ are*

$$\kappa_1 = \frac{\sqrt{c+3}}{2}, \quad \kappa_2 = \frac{1}{2}\sqrt{\frac{6(c-1)(5-c)}{c+3}}, \quad \kappa_3 = \frac{1}{2}\sqrt{\frac{3(c-1)(3c-7)}{c+3}}.$$

Moreover, the φ-torsions of γ are given by

$$\begin{cases} \tau_{12} = \mp\sqrt{\frac{2(5-c)}{c+3}}, \quad \tau_{13} = 0, \quad \tau_{14} = \pm\sqrt{\frac{3c-7}{c+3}}, \\ \tau_{23} = \mp\frac{3c-7}{\sqrt{3(c-1)(c+3)}}, \quad \tau_{24} = 0, \quad \tau_{34} = \pm\sqrt{\frac{2(5-c)(3c-7)}{3(c-1)(c+3)}}. \end{cases}$$

Proof. Let γ be a proper-biharmonic Legendre Frenet curve in $N(c)$ of osculating order $r = 4$. Then $c \neq 1$ and τ_{12} is different from 0, 1 or -1. From Theorem 3.10 we have $\varphi E_1 = \cos\alpha_0 E_2 + \sin\alpha_0 E_4$. It results that

$$\tau_{12} = -\cos\alpha_0, \quad \tau_{13} = 0, \quad \tau_{14} = -\sin\alpha_0, \quad \text{and} \quad \tau_{24} = 0.$$

In order to prove that τ_{23} is constant we differentiate the expression of φE_1 along γ and using the Frenet equations we obtain

$$\begin{aligned}\nabla_{E_1} \varphi E_1 &= \cos\alpha_0 \nabla_{E_1} E_2 + \sin\alpha_0 \nabla_{E_1} E_4 \\ &= -\kappa_1 \cos\alpha_0 E_1 + (\kappa_2 \cos\alpha_0 - \kappa_3 \sin\alpha_0) E_3.\end{aligned}$$

On the other hand, $\nabla_{E_1}\varphi E_1 = \kappa_1\varphi E_2 + \xi$ and therefore we have

$$\kappa_1\varphi E_2 + \xi = -\kappa_1 \cos\alpha_0 E_1 + (\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)E_3. \tag{3.8}$$

We take the scalar product in (3.8) with ξ and obtain

$$(\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)\eta(E_3) = 1. \tag{3.9}$$

In the same way as in the proof of Proposition 3.34 we get

$$\begin{aligned}\eta(E_3) &= g(E_3,\xi) = \frac{1}{\kappa_2}\Big(g(\nabla_{E_1}E_2,\xi) + \kappa_1 g(E_1,\xi)\Big) = -\frac{1}{\kappa_2}g(E_2,\nabla_{E_1}\xi)\\ &= -\frac{1}{\kappa_2}\tau_{12} = \frac{\cos\alpha_0}{\kappa_2}\end{aligned}$$

and then, from (3.9),

$$\kappa_2\sin\alpha_0 = -\kappa_3\cos\alpha_0.$$

Therefore $\alpha_0 \in (\frac{\pi}{2},\pi)\cup(\frac{3\pi}{2},2\pi)$.
Next, from Theorem 3.10, we have

$$\kappa_1^2 = \frac{c+3}{4},\quad \kappa_2^2 = \frac{3(c-1)}{4}\cos^2\alpha_0,\quad \kappa_3^2 = \frac{3(c-1)}{4}\sin^2\alpha_0,$$

and so c must be greater than 1.

Now, we take the scalar product in (3.8) with E_3, φE_2 and φE_4, respectively, and we get

$$\kappa_1\tau_{23} = -(\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0) + \eta(E_3) = -\frac{\kappa_2}{\cos\alpha_0} + \frac{\cos\alpha_0}{\kappa_2} \tag{3.10}$$

$$\kappa_1\sin^2\alpha_0 = -(\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)\tau_{23} = -\frac{\kappa_2}{\cos\alpha_0}\tau_{23} \tag{3.11}$$

$$0 = \kappa_1\cos\alpha_0\sin\alpha_0 + (\kappa_2\cos\alpha_0 - \kappa_3\sin\alpha_0)\tau_{34} = \kappa_1\cos\alpha_0\sin\alpha_0 + \frac{\kappa_2}{\cos\alpha_0}\tau_{34}. \tag{3.12}$$

and then, equations (3.10) and (3.11) lead to

$$\kappa_1^2\sin^2\alpha_0 = \frac{\kappa_2^2}{\cos^2\alpha_0} - 1.$$

We come to the conclusion $\sin^2\alpha_0 = \frac{3c-7}{c+3}$, so $c \in (\frac{7}{3},5)$, and then we obtain the expressions of the curvatures and the φ-torsions. □

Theorem 3.17 ([70]). *Let $\gamma : I \to \mathbb{S}^{2n+1}(1)$, $n \geq 2$, be a proper-biharmonic Legendre curve parametrized by arc-length. Then the equation of γ in the Euclidean space $\mathbb{R}^{2n+2}$, is either*

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}s\right)e_1 + \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}s\right)e_2 + \frac{1}{\sqrt{2}}e_3$$

where $\{e_i, \mathcal{J}e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to each other, or

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4,$$

where

$$A=\sqrt{1+\kappa_1},\quad B=\sqrt{1-\kappa_1},\quad \kappa_1\in(0,1), \tag{3.13}$$

and $\{e_i\}_{i=1}^4$ *are constant unit vectors orthogonal to each other, satisfying*

$$\langle e_1,\mathcal{J}e_3\rangle=\langle e_1,\mathcal{J}e_4\rangle=\langle e_2,\mathcal{J}e_3\rangle=\langle e_2,\mathcal{J}e_4\rangle=0,\quad A\langle e_1,\mathcal{J}e_2\rangle+B\langle e_3,\mathcal{J}e_4\rangle=0.$$

Proof. Let us denote by $\dot\nabla$ and by $\widetilde\nabla$ the Levi-Civita connections on $(\mathbb{S}^{2n+1},g_0)$ and $(\mathbb{R}^{2n+2},\langle,\rangle)$, respectively.

First, assume that γ is the biharmonic circle, that is $\kappa_1=1$. From the Gauss and Frenet equations we get

$$\widetilde\nabla_T T=\dot\nabla_T T-\langle T,T\rangle\gamma=\kappa_1E_2-\gamma$$

and

$$\widetilde\nabla_T\widetilde\nabla_T T=(-\kappa_1^2-1)T=-2T,$$

which implies

$$\gamma'''+2\gamma'=0.$$

The general solution of the above equation is

$$\gamma(s)=\cos\left(\sqrt2 s\right)c_1+\sin\left(\sqrt2 s\right)c_2+c_3,$$

where $\{c_i\}$ are constant vectors in $\mathbb{R}^{2n+2}$.
Now, as γ satisfies

$$\langle\gamma,\gamma\rangle=1,\ \langle\gamma',\gamma'\rangle=1,\ \langle\gamma,\gamma'\rangle=0,\ \langle\gamma',\gamma''\rangle=0,\ \langle\gamma'',\gamma''\rangle=2,\ \langle\gamma,\gamma''\rangle=-1,$$

and since in $s=0$ we have $\gamma=c_1+c_3$, $\gamma'=\sqrt2c_2$, $\gamma''=-2c_1$, we obtain

$$c_{11}+2c_{13}+c_{33}=1,\ c_{22}=\frac12,\ c_{12}+c_{23}=0,\ c_{12}=0,\ c_{11}=\frac12, c_{11}+c_{13}=\frac12,$$

where c_{ij} denotes $\langle c_i,c_j\rangle$. The above relations imply that $\{c_i\}$ are orthogonal vectors in $\mathbb{R}^{2n+2}$ with $|c_1|=|c_2|=|c_3|=\frac{1}{\sqrt2}$.
Finally, using the fact that γ is a Legendre curve one obtains easily that $\langle c_i,\mathcal{J}c_j\rangle=0$ for any $i,j=1,2,3$. If we denote $e_i=\sqrt2c_i$ we obtain the first part of the Theorem.

Suppose now γ is the biharmonic helix, that is $\kappa_1^2+\kappa_2^2=1$, $\kappa_1\in(0,1)$. From the Gauss and Frenet equations we get

$$\widetilde\nabla_T T=\dot\nabla_T T-\langle T,T\rangle\gamma=\kappa_1E_2-\gamma,$$

$$\widetilde\nabla_T\widetilde\nabla_T T=\kappa_1\widetilde\nabla_T E_2-T=\kappa_1\Big(-\kappa_1T+\kappa_2E_3\Big)-T=-\Big(\kappa_1^2+1\Big)T+\kappa_1\kappa_2E_3$$

and

$$\widetilde\nabla_T\widetilde\nabla_T\widetilde\nabla_T T=-\Big(\kappa_1^2+1\Big)\widetilde\nabla_T T+\kappa_1\kappa_2\widetilde\nabla_T E_3=-\Big(\kappa_1^2+1\Big)\widetilde\nabla_T T-\kappa_1\kappa_2^2E_2=-2\gamma''-\kappa_2^2\gamma.$$

Hence

$$\gamma^{iv}+2\gamma''+\kappa_2^2\gamma=0,$$

and its general solution is

$$\gamma(s) = \cos(As)c_1 + \sin(As)c_2 + \cos(Bs)c_3 + \sin(Bs)c_4,$$

where A, B are given by (3.13) and $\{c_i\}$ are constant vectors in $\mathbb{R}^{2n+2}$.
As γ satisfies

$$\langle\gamma,\gamma\rangle = 1,\ \langle\gamma',\gamma'\rangle = 1,\ \langle\gamma,\gamma'\rangle = 0,\ \langle\gamma',\gamma''\rangle = 0,\ \langle\gamma'',\gamma''\rangle = 1+\kappa_1^2,$$

$$\langle\gamma,\gamma''\rangle = -1,\ \langle\gamma',\gamma'''\rangle = -(1+\kappa_1^2),\ \langle\gamma'',\gamma'''\rangle = 0,\ \langle\gamma,\gamma'''\rangle = 0,\ \langle\gamma''',\gamma'''\rangle = 3\kappa_1^2+1,$$

and since in $s = 0$ we have $\gamma = c_1 + c_3$, $\gamma' = Ac_2 + Bc_4$, $\gamma'' = -A^2c_1 - B^2c_3$, $\gamma''' = -A^3c_2 - B^3c_4$, we obtain

$$c_{11} + 2c_{13} + c_{33} = 1 \tag{3.14}$$

$$A^2c_{22} + 2ABc_{24} + B^2c_{44} = 1 \tag{3.15}$$

$$Ac_{12} + Ac_{23} + Bc_{14} + Bc_{34} = 0 \tag{3.16}$$

$$A^3c_{12} + AB^2c_{23} + A^2Bc_{14} + B^3c_{34} = 0 \tag{3.17}$$

$$A^4c_{11} + 2A^2B^2c_{13} + B^4c_{33} = 1+\kappa_1^2 \tag{3.18}$$

$$A^2c_{11} + (A^2+B^2)c_{13} + B^2c_{33} = 1 \tag{3.19}$$

$$A^4c_{22} + (AB^3+A^3B)c_{24} + B^4c_{44} = 1+\kappa_1^2 \tag{3.20}$$

$$A^5c_{12} + A^3B^2c_{23} + A^2B^3c_{14} + B^5c_{34} = 0 \tag{3.21}$$

$$A^3c_{12} + A^3c_{23} + B^3c_{14} + B^3c_{34} = 0 \tag{3.22}$$

$$A^6c_{22} + 2A^3B^3c_{24} + B^6c_{44} = 3\kappa_1^2+1 \tag{3.23}$$

where $c_{ij} = \langle c_i, c_j\rangle$. Since the determinant of the system given by (3.16), (3.17), (3.21) and (3.22) is $-A^2B^2(A^2-B^2)^4 \neq 0$ it follows that

$$c_{12} = c_{23} = c_{14} = c_{34} = 0.$$

The equations (3.14), (3.18) and (3.19) give

$$c_{11} = \frac{1}{2},\quad c_{13} = 0,\quad c_{33} = \frac{1}{2},$$

and, from (3.15), (3.20) and (3.23) follows that

$$c_{22} = \frac{1}{2},\quad c_{24} = 0,\quad c_{44} = \frac{1}{2}.$$

Therefore, we obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{R}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = |c_4| = \frac{1}{\sqrt{2}}$.

Finally, since γ is a Legendre curve one obtains the second part of the Theorem. □

Remark 3.18 ([70]). We note that if γ is a proper-biharmonic Legendre circle, then $E_2 \perp \varphi T$ and $n \geq 3$. If γ is a proper-biharmonic Legendre helix, then $g_0(E_2, \varphi T) = -A\langle e_1, \mathcal{J} e_2\rangle$ and we have two cases: either $E_2 \perp \varphi T$ and then $\{e_i, \mathcal{J} e_j\}_{i,j=1}^4$ is an orthonormal system in $\mathbb{R}^{2n+2}$, so $n \geq 3$, or $g_0(E_2, \varphi T) \neq 0$ and, in this case, $g_0(E_2, \varphi T) \in (-1,1)\setminus\{0\}$. We also observe that φT cannot be parallel to E_2. When $g_0(E_2, \varphi T) \neq 0$ and $n \geq 3$ the first four vectors (for example) in the canonical basis of the Euclidean space $\mathbb{R}^{2n+2}$ satisfy the conditions of Theorem 3.17, whilst for $n = 2$ we can obtain four vectors $\{e_1, e_2, e_3, e_4\}$ satisfying these conditions in the following way. We consider constant unit vectors e_1, e_3 and f in $\mathbb{R}^6$ such that $\{e_1, e_3, f, \mathcal{J} e_1, \mathcal{J} e_3, \mathcal{J} f\}$ is a $\mathcal{J}$-basis. Then, by a straightforward computation, it follows that the vectors e_2 and e_4 have to be given by

$$e_2 = \mp\frac{B}{A}\mathcal{J} e_1 + \alpha_1 f + \alpha_2 \mathcal{J} f, \quad e_4 = \pm\mathcal{J} e_3,$$

where α_1 and α_2 are constants such that $\alpha_1^2 + \alpha_2^2 = 1 - B^2/A^2 = 2\kappa_1/A^2$. As a concrete example, we can start with the following vectors in $\mathbb{R}^6$:

$$e_1 = (1,0,0,0,0,0), \quad e_3 = (0,0,1,0,0,0), \quad f = (0,1,0,0,0,0)$$

and obtain

$$e_2 = \Big(0, \alpha_1, 0, -\frac{B}{A}, \alpha_2, 0\Big), \quad e_4 = (0,0,0,0,0,1),$$

where $\alpha_1^2 + \alpha_2^2 = 1 - B^2/A^2$.

The classification of all proper-biharmonic Legendre curves in a Sasakian space form $N^{2n+1}(c)$ was given in [70]. This classification is invariant under an isometry Ψ of N which preserves ξ (or, equivalently, Ψ is φ-holomorphic).

In order to find higher dimensional proper-biharmonic submanifolds in a Sasakian space form we gave Theorem 3.22.

Next we shall use the deformed Sasakian structure (φ, ξ, η, g) on $\mathbb{S}^{2n+1}$.

Theorem 3.19 ([70]). *Let $\gamma : I \to \mathbb{S}^{2n+1}(c)$, $n \geq 2$, $c > -3$ and $c \neq 1$, be a proper-biharmonic Legendre curve parametrized by arc-length such that $E_2 \perp \varphi T$. Then the equation of γ in the Euclidean space $\mathbb{R}^{2n+2}$ is either*

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos\Big(\sqrt{\frac{2}{a}}s\Big)e_1 + \frac{1}{\sqrt{2}}\sin\Big(\sqrt{\frac{2}{a}}s\Big)e_2 + \frac{1}{\sqrt{2}}e_3,$$

for $n \geq 2$, where $\{e_i, \mathcal{J} e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to each other, or

$$\gamma(s) = \frac{1}{\sqrt{2}}\cos(As)e_1 + \frac{1}{\sqrt{2}}\sin(As)e_2 + \frac{1}{\sqrt{2}}\cos(Bs)e_3 + \frac{1}{\sqrt{2}}\sin(Bs)e_4,$$

for $n \geq 3$, where

$$A = \sqrt{\frac{1+\kappa_1\sqrt{a}}{a}}, \quad B = \sqrt{\frac{1-\kappa_1\sqrt{a}}{a}}, \quad \kappa_1 \in \Big(0, \frac{1}{a}\Big), \tag{3.24}$$

and $\{e_i, \mathcal{J} e_j\}_{i,j=1}^3$ are constant unit vectors orthogonal to each other.

Proof. Again let us denote by ∇, $\dot{\nabla}$ and by $\widetilde{\nabla}$ the Levi-Civita connections on $(\mathbb{S}^{2n+1}, g)$, $(\mathbb{S}^{2n+1}, g_0)$ and $(\mathbb{R}^{2n+2}, \langle , \rangle)$, respectively. From the definition of the Levi-Civita connection, as $g_0(X, \varphi_0 Y) = d\eta_0(X, Y)$ and $g(X, \varphi Y) = d\eta(X, Y)$, we obtain $g(\nabla_X Y, Z) = a g_0(\dot{\nabla}_X Y, Z)$, for any vector field Z and for any X, Y which satisfy $X \perp \xi$, $Y \perp \xi$ and $X \perp \varphi Y$. Further, it is easy to check that we have

$$\nabla_X Y = \dot{\nabla}_X Y, \quad \forall X, Y \in C(T\mathbb{S}^{2n+1}) \text{ with } X \perp \xi,\ Y \perp \xi,\ X \perp \varphi Y. \tag{3.25}$$

First we consider the case when γ is the biharmonic circle, that is $\kappa_1^2 = \frac{c+3}{4}$. Let $T = \gamma'$ be the unit tangent vector field (with respect to the metric g) along γ. Using (3.25) we obtain $\dot{\nabla}_T T = \nabla_T T$ and $\dot{\nabla}_T E_2 = \nabla_T E_2$.
From the Gauss and Frenet equations we get

$$\widetilde{\nabla}_T T = \dot{\nabla}_T T - \langle T, T\rangle \gamma = \kappa_1 E_2 - \frac{1}{a}\gamma$$

and

$$\widetilde{\nabla}_T \widetilde{\nabla}_T T = (-\kappa_1^2 - \frac{1}{a})T = -\frac{2}{a}T.$$

Hence

$$a\gamma''' + 2\gamma' = 0,$$

with the general solution

$$\gamma(s) = \cos\Big(\sqrt{\frac{2}{a}}s\Big)c_1 + \sin\Big(\sqrt{\frac{2}{a}}s\Big)c_2 + c_3,$$

where $\{c_i\}$ are constant vectors in $\mathbb{R}^{2n+2}$.
As γ verifies the following equations,

$$\langle \gamma, \gamma\rangle = 1,\ \langle \gamma', \gamma'\rangle = \frac{1}{a},\ \langle \gamma, \gamma'\rangle = 0,\ \langle \gamma', \gamma''\rangle = 0,\ \langle \gamma'', \gamma''\rangle = \frac{2}{a^2},\ \langle \gamma, \gamma''\rangle = -\frac{1}{a},$$

and in $s = 0$ we have $\gamma = c_1 + c_3$, $\gamma' = \sqrt{\frac{2}{a}}c_2$, $\gamma'' = -\frac{2}{a}c_1$, one obtains

$$c_{11} + 2c_{13} + c_{33} = 1,\ c_{22} = \frac{1}{2},\ c_{12} + c_{23} = 0,\ c_{12} = 0,\ c_{11} = \frac{1}{2}, c_{11} + c_{13} = \frac{1}{2},$$

where $c_{ij} = \langle c_i, c_j\rangle$. Consequently, we obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{R}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = \frac{1}{\sqrt{2}}$.
Finally, using the facts that γ is a Legendre curve and $g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0$ one obtains easily that $\langle c_i, \mathcal{J}c_j\rangle = 0$ for any $i, j = 1, 2, 3$.

Now we assume that γ is a biharmonic helix, that is $\kappa_1^2 + \kappa_2^2 = \frac{c+3}{4}$, $\kappa_1^2 \in \Big(0, \frac{c+3}{4}\Big)$. First, using (3.25), we obtain $\dot{\nabla}_T T = \nabla_T T$, $\dot{\nabla}_T E_2 = \nabla_T E_2$ and $\dot{\nabla}_T E_3 = \nabla_T E_3$.
From the Gauss and Frenet equations we get

$$\widetilde{\nabla}_T T = \dot{\nabla}_T T - \langle T, T\rangle \gamma = \kappa_1 E_2 - \frac{1}{a}\gamma,$$

$$\widetilde{\nabla}_T \widetilde{\nabla}_T T = \kappa_1 \widetilde{\nabla}_T E_2 - \frac{1}{a}T = \kappa_1\Big(-\kappa_1 T + \kappa_2 E_3\Big) - \frac{1}{a}T = -\Big(\kappa_1^2 + \frac{1}{a}\Big)T + \kappa_1\kappa_2 E_3,$$

and

$$\begin{aligned}\widetilde{\nabla}_T\widetilde{\nabla}_T\widetilde{\nabla}_T T &= -\Big(\kappa_1^2+\tfrac{1}{a}\Big)\widetilde{\nabla}_T T+\kappa_1\kappa_2\widetilde{\nabla}_T E_3=-\Big(\kappa_1^2+\tfrac{1}{a}\Big)\widetilde{\nabla}_T T-\kappa_1\kappa_2^2E_2\\ &= -\tfrac{2}{a}\gamma''-\tfrac{1}{a}\kappa_2^2\gamma.\end{aligned}$$

Therefore

$$a\gamma^{iv}+2\gamma''+\kappa_2^2\gamma=0,$$

and its general solution is

$$\gamma(s)=\cos(As)c_1+\sin(As)c_2+\cos(Bs)c_3+\sin(Bs)c_4,$$

where A, B are given by (3.24) and $\{c_i\}$ are constant vectors in $\mathbb{R}^{2n+2}$.
The curve γ satisfies

$$\langle\gamma,\gamma\rangle=1,\ \langle\gamma',\gamma'\rangle=\frac{1}{a},\ \langle\gamma,\gamma'\rangle=0,\ \langle\gamma',\gamma''\rangle=0,\ \langle\gamma'',\gamma''\rangle=\frac{1+a\kappa_1^2}{a^2},$$

$$\langle\gamma,\gamma''\rangle=-\frac{1}{a},\ \langle\gamma',\gamma'''\rangle=-\frac{1+a\kappa_1^2}{a^2},\ \langle\gamma'',\gamma'''\rangle=0,\ \langle\gamma,\gamma'''\rangle=0,$$

$$\langle\gamma''',\gamma'''\rangle=\frac{3a\kappa_1^2+1}{a^3},$$

and in $s=0$ we have

$$\gamma=c_1+c_3,\quad \gamma'=Ac_2+Bc_4,\quad \gamma''=-A^2c_1-B^2c_3,\quad \gamma'''=-A^3c_2-B^3c_4.$$

Then, it follows

$$c_{11}+2c_{13}+c_{33}=1 \tag{3.26}$$

$$A^2c_{22}+2ABc_{24}+B^2c_{44}=\frac{1}{a} \tag{3.27}$$

$$Ac_{12}+Ac_{23}+Bc_{14}+Bc_{34}=0 \tag{3.28}$$

$$A^3c_{12}+AB^2c_{23}+A^2Bc_{14}+B^3c_{34}=0 \tag{3.29}$$

$$A^4c_{11}+2A^2B^2c_{13}+B^4c_{33}=\frac{1+a\kappa_1^2}{a^2} \tag{3.30}$$

$$A^2c_{11}+(A^2+B^2)c_{13}+B^2c_{33}=\frac{1}{a} \tag{3.31}$$

$$A^4c_{22}+(AB^3+A^3B)c_{24}+B^4c_{44}=\frac{1+a\kappa_1^2}{a^2} \tag{3.32}$$

$$A^5c_{12}+A^3B^2c_{23}+A^2B^3c_{14}+B^5c_{34}=0 \tag{3.33}$$

$$A^3c_{12}+A^3c_{23}+B^3c_{14}+B^3c_{34}=0 \tag{3.34}$$

$$A^6c_{22}+2A^3B^3c_{24}+B^6c_{44}=\frac{3a\kappa_1^2+1}{a^3} \tag{3.35}$$

where $c_{ij}=\langle c_i,c_j\rangle$.

The solution of the system given by (3.28), (3.29), (3.33) and (3.34) is

$$c_{12} = c_{23} = c_{14} = c_{34} = 0.$$

From equations (3.26), (3.30) and (3.31) we get

$$c_{11} = \frac{1}{2}, \quad c_{13} = 0, \quad c_{33} = \frac{1}{2},$$

and, from (3.27), (3.32), (3.35),

$$c_{22} = \frac{1}{2}, \quad c_{24} = 0, \quad c_{44} = \frac{1}{2}.$$

We obtain that $\{c_i\}$ are orthogonal vectors in $\mathbb{R}^{2n+2}$ with $|c_1| = |c_2| = |c_3| = |c_4| = \frac{1}{\sqrt{2}}$. Finally, since γ is a Legendre curve and $g(\nabla_{\gamma'}\gamma', \varphi\gamma') = 0$, one obtains the conclusion. □

In the third case, just like for $\mathbb{S}^3$ (see [71]), we obtain.

Theorem 3.20 ([70]). *Let $\gamma : I \to \mathbb{S}^{2n+1}(c)$, $c > 1$, be a proper-biharmonic Legendre curve parametrized by arc-length such that $E_2 \parallel \varphi T$. Then the equation of γ in the Euclidean space $\mathbb{R}^{2n+2}$ is*

$$\begin{aligned}\gamma(s) &= \sqrt{\tfrac{B}{A+B}}\cos(As)e_1 - \sqrt{\tfrac{B}{A+B}}\sin(As)\mathcal{J}e_1\\ &\quad+\sqrt{\tfrac{A}{A+B}}\cos(Bs)e_3 + \sqrt{\tfrac{A}{A+B}}\sin(Bs)\mathcal{J}e_3\\ &= \sqrt{\tfrac{B}{A+B}}\exp(-\mathrm{i}As)e_1 + \sqrt{\tfrac{A}{A+B}}\exp(\mathrm{i}Bs)e_3,\end{aligned}$$

where $\{e_1, e_3\}$ are constant unit orthogonal vectors in $\mathbb{R}^{2n+2}$ with e_3 orthogonal to $\mathcal{J}e_1$, and

$$A = \sqrt{\frac{3-2a-2\sqrt{(a-1)(a-2)}}{a}}, \quad B = \sqrt{\frac{3-2a+2\sqrt{(a-1)(a-2)}}{a}}. \tag{3.36}$$

Remark 3.21 ([70]). For the fourth case the ODE satisfied by proper-biharmonic Legendre curves in the unit $(2n+1)$-sphere may be also obtained but the computations are rather complicated.

3.1.4 Biharmonic submanifolds in Sasakian space forms

A method to obtain biharmonic submanifolds in a Sasakian space form is provided by the following Theorem.

Theorem 3.22 ([70]). *Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c and let $\mathbf{i} : M \to N$ be an r-dimensional integral submanifold of N, $1 \le r \le n$. Consider*

$$F : \widetilde{M} = I \times M \to N, \quad F(t,p) = \phi_t(p) = \phi_p(t),$$

where $I=\mathbb{S}^1$ or $I=\mathbb{R}$ and $\{\phi_t\}_{t\in I}$ is the flow of the vector field ξ. Then $F:(\widetilde{M},\widetilde{g}=dt^2+\mathbf{i}^*g)\to N$ is a Riemannian immersion, and it is proper-biharmonic if and only if M is a proper-biharmonic submanifold of N.

Proof. From the definition of the flow of ξ we have

$$dF(t,p)\Big(\frac{\partial}{\partial t}\Big)=\frac{d}{ds}|_{s=t}\{\phi_p(s)\}=\dot{\phi}_p(t)=\xi(\phi_p(t))=\xi(F(t,p)),$$

i.e. $\frac{\partial}{\partial t}$ is F-correlated to ξ and

$$\Big|dF(t,p)\Big(\frac{\partial}{\partial t}\Big)\Big|=|\xi(F(t,p))|=1=\Big|\frac{\partial}{\partial t}\Big|.$$

The vector $X_p\in T_pM$ can be identified to $(0,X_p)\in T_{(t,p)}(I\times M)$ and we have

$$dF_{(t,p)}(X_p)=(dF)_{(t,p)}(\dot{\gamma}(0))=\frac{d}{ds}|_{s=0}\{\phi_t(\gamma(s))\}=(d\phi_t)_p(X_p).$$

Since ϕ_t is an isometry $|dF_{(t,p)}(X_p)|=|(d\phi_t)_p(X_p)|=|X_p|$.
Moreover,

$$\begin{aligned} g\Big(dF_{(t,p)}\Big(\tfrac{\partial}{\partial t}\Big),dF_{(t,p)}(X_p)\Big) &= g(\xi(\phi_p(t)),(d\phi_t)_p(X_p)) \\ &= g((d\phi_t)_p(\xi_p),(d\phi_t)_p(X_p))=g(\xi_p,X_p)=0 \\ &= \widetilde{g}(\tfrac{\partial}{\partial t},X_p), \end{aligned}$$

and therefore $F:(I\times M,\widetilde{g})\to N$ is a Riemannian immersion.

Let $F^{-1}(TN)$ be the pull-back bundle over $\widetilde{M}$ and ∇^F the pull-back connection determined by the Levi-Civita connection on N. We shall prove that

$$\tau(F)_{(t,p)}=(d\phi_t)_p(\tau(\mathbf{i}))\quad\text{and}\quad\tau_2(F)_{(t,p)}=(d\phi_t)_p(\tau_2(\mathbf{i})),$$

so, from the point of view of harmonicity and biharmonicity, $\widetilde{M}$ and M have the same behaviour.

We start with two remarks. First, let $\sigma\in C(F^{-1}(TN))$ be a section in $F^{-1}(TN)$ defined by $\sigma_{(t,p)}=(d\phi_t)_p(Z_p)$, where Z is a vector field along M, i.e. $Z_p\in T_pN$, $\forall p\in M$. One can easily check that

$$(\nabla^F_X\sigma)_{(t,p)}=(d\phi_t)_p(\nabla^N_X Z),\quad\forall X\in C(TM).\tag{3.37}$$

Then, if $\sigma\in C(F^{-1}(TN))$, it follows that $\varphi\sigma$ given by $(\varphi\sigma)_{(t,p)}=\varphi_{\phi_p(t)}(\sigma_{(t,p)})$ is a section in $F^{-1}(TN)$ and

$$\nabla^F_{\frac{\partial}{\partial t}}\varphi\sigma=\varphi\nabla^F_{\frac{\partial}{\partial t}}\sigma.\tag{3.38}$$

Now, we consider $\{X_1,...,X_r\}$ a local orthonormal frame field on U, where U is an open subset of M. The tension field of F is given by

$$\tau(F)=\nabla^F_{\frac{\partial}{\partial t}}dF\Big(\frac{\partial}{\partial t}\Big)-dF\Big(\nabla^{\widetilde{M}}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}\Big)+\sum_{a=1}^r\{\nabla^F_{X_a}dF(X_a)-dF(\nabla^{\widetilde{M}}_{X_a}X_a)\}.\tag{3.39}$$

As

$$\nabla^F_{\frac{\partial}{\partial t}} dF\Big(\frac{\partial}{\partial t}\Big) = \nabla^N_\xi \xi = 0, \quad \nabla^{\widetilde{M}}_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \nabla^I_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0,$$

$$(\nabla^F_{X_a} dF(X_a))_{(t,p)} = (d\phi_t)_p(\nabla^N_{X_a} X_a), \quad dF_{(t,p)}(\nabla^{\widetilde{M}}_{X_a} X_a) = (d\phi_t)_p(\nabla^M_{X_a} X_a),$$

replacing in (3.39) we get

$$\tau(F)_{(t,p)} = (d\phi_t)_p(\tau(\mathbf{i})).$$

In order to obtain that $\tau_2(F)_{(t,p)} = (d\phi_t)_p(\tau_2(\mathbf{i}))$, we shall prove first that $\nabla^F_{\frac{\partial}{\partial t}} \tau(F) = -\varphi(\tau(F))$.

Since $[\frac{\partial}{\partial t}, X_a] = 0$, $a = 1, ..., r$, it follows that

$$\nabla^F_{\frac{\partial}{\partial t}} dF(X_a) = \nabla^F_{X_a} dF\Big(\frac{\partial}{\partial t}\Big).$$

But

$$\begin{aligned} \Big(\nabla^F_{X_a} dF\Big(\tfrac{\partial}{\partial t}\Big)\Big)_{(t,p)} &= \nabla^N_{dF_{(t,p)}X_a} \xi = \nabla^N_{(d\phi_t)_p X_a} \xi = -\varphi((d\phi_t)_p(X_a)) \\ &= -(d\phi_t)_p(\varphi X_a), \end{aligned}$$

so

$$\Big(\nabla^F_{\frac{\partial}{\partial t}} dF(X_a)\Big)_{(t,p)} = -(d\phi_t)_p(\varphi X_a). \tag{3.40}$$

We note that

$$R^F\Big(\frac{\partial}{\partial t}, X_a\Big) dF(X_a) = \nabla^F_{\frac{\partial}{\partial t}} \nabla^F_{X_a} dF(X_a) - \nabla^F_{X_a} \nabla^F_{\frac{\partial}{\partial t}} dF(X_a)$$

and, on the other hand, as N is a Sasakian space form,

$$\Big(R^F\Big(\frac{\partial}{\partial t}, X_a\Big) dF(X_a)\Big)_{(t,p)} = R^N_{\phi_t(p)}(\xi, (d\phi_t)_p(X_a))(d\phi_t)_p(X_a) = \xi.$$

Therefore

$$\nabla^F_{\frac{\partial}{\partial t}} \nabla^F_{X_a} dF(X_a) - \nabla^F_{X_a} \nabla^F_{\frac{\partial}{\partial t}} dF(X_a) = \xi \tag{3.41}$$

Using (3.37) and (3.40), $\nabla^F_{X_a} \nabla^F_{\frac{\partial}{\partial t}} dF(X_a)$ can be written as

$$\begin{aligned} \Big(\nabla^F_{X_a} \nabla^F_{\frac{\partial}{\partial t}} dF(X_a)\Big)_{(t,p)} &= -(d\phi_t)_p(\nabla^N_{X_a} \varphi X_a) \\ &= -(d\phi_t)_p(\xi + \varphi \nabla^N_{X_a} X_a). \end{aligned} \tag{3.42}$$

Moreover, from (3.40)

$$\begin{aligned} \Big(\nabla^F_{\frac{\partial}{\partial t}} dF(\nabla^{\widetilde{M}}_{X_a} X_a)\Big)_{(t,p)} &= \Big(\nabla^F_{\frac{\partial}{\partial t}} dF(\nabla^M_{X_a} X_a)\Big)_{(t,p)} \\ &= -(d\phi_t)_p(\varphi \nabla^M_{X_a} X_a). \end{aligned} \tag{3.43}$$

Replacing (3.42) in (3.41) and using (3.43), we obtain

$$\begin{aligned}\xi &= \nabla^F_{\frac{\partial}{\partial t}}\nabla^F_{X_a}dF(X_a) - \nabla^F_{\frac{\partial}{\partial t}}dF(\nabla^{\widetilde{M}}_{X_a}X_a) + \nabla^F_{\frac{\partial}{\partial t}}dF(\nabla^{\widetilde{M}}_{X_a}X_a) - \nabla^F_{X_a}\nabla^F_{\frac{\partial}{\partial t}}dF(X_a)\\ &= \nabla^F_{\frac{\partial}{\partial t}}\nabla dF(X_a,X_a) - (d\phi_t)_p(\varphi\nabla^M_{X_a}X_a) + (d\phi_t)_p(\xi + \varphi\nabla^N_{X_a}X_a)\\ &= \nabla^F_{\frac{\partial}{\partial t}}\nabla dF(X_a,X_a) + \varphi(d\phi_t)_p(\nabla^N_{X_a}X_a - \nabla^M_{X_a}X_a) + \xi,\end{aligned}$$

so

$$\Big(\nabla^F_{\frac{\partial}{\partial t}}\nabla dF(X_a,X_a)\Big)_{(t,p)} = -\varphi(d\phi_t)_p(\nabla d\mathbf{i}(X_a,X_a)). \tag{3.44}$$

Since $\nabla dF(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = 0$, summing up in (3.44) we obtain

$$\nabla^F_{\frac{\partial}{\partial t}}\tau(F) = -\varphi(\tau(F)). \tag{3.45}$$

From (3.38) and (3.45) we have

$$\begin{aligned}\nabla^F_{\frac{\partial}{\partial t}}\nabla^F_{\frac{\partial}{\partial t}}\tau(F) &= -\nabla^F_{\frac{\partial}{\partial t}}\varphi(\tau(F)) = -\varphi\nabla^F_{\frac{\partial}{\partial t}}\tau(F) = \varphi^2\tau(F)\\ &= -\tau(F),\end{aligned} \tag{3.46}$$

and from (3.37)

$$(\nabla^F_{X_a}\nabla^F_{X_a}\tau(F))_{(t,p)} = (d\phi_t)_p(\nabla^N_{X_a}\nabla^N_{X_a}\tau(\mathbf{i})), \tag{3.47}$$

$$\Big(\nabla^F_{\nabla^{\widetilde{M}}_{X_a}X_a}\tau(F)\Big)_{(t,p)} = (d\phi_t)_p\Big(\nabla^N_{\nabla^M_{X_a}X_a}\tau(\mathbf{i})\Big). \tag{3.48}$$

From (3.46), (3.47) and (3.48) we obtain

$$\begin{aligned}-(\Delta^F\tau(F))_{(t,p)} &= \nabla^F_{\frac{\partial}{\partial t}}\nabla^F_{\frac{\partial}{\partial t}}\tau(F) + \textstyle\sum_{a=1}^r\{\nabla^F_{X_a}\nabla^F_{X_a}\tau(F) - \nabla^F_{\nabla^{\widetilde{M}}_{X_a}X_a}\tau(F)\}\\ &= -\tau(F)_{(t,p)} - (d\phi_t)_p(\Delta^{\mathbf{i}}\tau(\mathbf{i})).\end{aligned} \tag{3.49}$$

Using the form of the curvature tensor field R^N, after a straightforward computation, we get

$$\operatorname{trace} R^F(dF,\tau(F))dF = -\tau(F) + (d\phi_t)_p(\operatorname{trace} R^N_p(d\mathbf{i},\tau(\mathbf{i}))d\mathbf{i}). \tag{3.50}$$

Finally, from (3.49) and (3.50) we conclude

$$\tau_2(F)_{(t,p)} = (d\phi_t)_p(\tau_2(\mathbf{i})).$$

□

Remark 3.23 ([70])**.** The previous result was expected because of the following remark. Assume that $(N^{2n+1},\varphi,\xi,\eta,g)$ is a compact strictly regular Sasakian manifold and let $G: M \to N$ be an arbitrary smooth map from a compact Riemannian manifold M. If F is biharmonic, then the map G is biharmonic, where $F: \widetilde{M} = \mathbb{S}^1 \times M \to N$, $F(t,p) = \phi_t(G(p))$.

Indeed, an arbitrary variation $\{G_s\}_s$ of G induces a variation $\{F_s\}_s$ of F. We can check that $\tau_{(p,t)}(F_s) = (d\phi_t)_{G_s(p)}(\tau_p(G_s))$ and, from the biharmonicity of F and the Fubini Theorem, we get

$$\begin{aligned} 0 &= \frac{d}{ds}|_{s=0}\{E_2(F_s)\} = \frac{1}{2}\frac{d}{ds}|_{s=0}\int_{\widetilde{M}} |\tau(F_s)|^2\ v_{\widetilde{g}} = \frac{1}{2}2\pi\frac{d}{ds}|_{s=0}\int_M |\tau(G_s)|^2\ v_g \\ &= 2\pi\frac{d}{ds}|_{s=0}\{E_2(G_s)\}. \end{aligned}$$

Since $\frac{d}{ds}|_{s=0}\{E_2(G_s)\} = 0$ for any variation $\{G_s\}_s$ of G, it follows that G is biharmonic. In particular, if M is a submanifold of N and G is the inclusion map $\mathbf{i}$, then we have the direct implication of the Theorem.

Theorem 3.24 ([70]). *Let $\widetilde{M}^2$ be a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then M is proper-biharmonic if and only if, locally, it is given by $F(t,s) = \phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve.*

Proof. A surface $\widetilde{M}$ of N^{2n+1} invariant under the flow-action of the characteristic vector field ξ, that is $\phi_t(p) \in M$, for any t and any $p \in M$, can be written, locally, $F(t,s) = \phi_t(\gamma(s))$, where γ is a Legendre curve in N. Then, from Theorem 3.22, such a surface is proper-biharmonic if and only if γ is proper-biharmonic. □

Corollary 3.25 ([70]). *Let $\widetilde{M}^2$ be a surface of $\mathbb{S}^{2n+1}$ endowed with its canonical Sasakian structure which is invariant under the flow-action of the characteristic vector field ξ. Then M is proper-biharmonic if and only if, locally, it is given by $F(t,s) = \phi_t(\gamma(s))$, where γ is a proper-biharmonic Legendre curve given by Theorem 3.17.*

Next, consider the unit $(2n+1)$-dimensional sphere $\mathbb{S}^{2n+1}$ endowed with its canonical or deformed Sasakian structure. The flow of ξ is $\phi_t(z) = \exp(-\mathrm{i}\frac{t}{a})z$, and from Theorems 3.19, 3.20 and 3.22 we obtain explicit examples of proper-biharmonic surfaces in $(\mathbb{S}^{2n+1}, \varphi, \xi, \eta, g)$, $a > 0$, of constant mean curvature.
Moreover, we reobtain a result in [4].

Proposition 3.26 ([4]). *Let $F : \widetilde{M}^3 \to (\mathbb{S}^5, \varphi_0, \xi_0, \eta_0, g_0) \subset \mathbb{R}^6$ be a proper-biharmonic anti-invariant immersion. Then*

$$F(t,u,v) = \frac{\exp(-\mathrm{i}t)}{\sqrt{2}}(\exp(\mathrm{i}u), \mathrm{i}\exp(-\mathrm{i}u)\sin(\sqrt{2}v), \mathrm{i}\exp(-\mathrm{i}u)\cos(\sqrt{2}v)).$$

Proof. It was proved in [120] that the proper-biharmonic integral surface of $(\mathbb{S}^5, \varphi_0, \xi_0, \eta_0, g_0)$ is given by

$$x(u,v) = \frac{1}{\sqrt{2}}(\exp(\mathrm{i}u), \mathrm{i}\exp(-\mathrm{i}u)\sin(\sqrt{2}v), \mathrm{i}\exp(-\mathrm{i}u)\cos(\sqrt{2}v)).$$

Now, composing with the flow of ξ_0 we reobtain the result in [4]. □

3.2 Biharmonic hypersurfaces in Sasakian space forms

3.2.1 Introduction

The first result of the second section is the characterization of the biharmonic submanifolds in a strictly regular Sasakian space form $N(c)$ obtained from submanifolds in the quotient space $\bar{N}(c+3)$ by using the Boothby-Wang fibration. We call such submanifolds Hopf cylinders. In order to insure the existence, we show that $c+3$ must be positive and then, by using the Takagi classification, we obtain all proper-biharmonic Hopf cylinders over homogeneous real hypersurfaces in complex projective spaces of constant holomorphic sectional curvature $c+3>0$.

3.2.2 Biharmonic hypersurfaces in Sasakian space forms

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a strictly regular Sasakian space form with constant φ-sectional curvature c, and $\pi : N \to \bar{N} = N/\xi$ the Boothby-Wang fibration. Let $\bar{\mathbf{i}} : \bar{M} \to \bar{N}$ be a submanifold and consider the associated Hopf cylinder $\mathbf{i} : M = \pi^{-1}(\bar{M}) \to N$, of dimension m. We shall denote by B, A and H the second fundamental form of M in N, the shape operator and the mean curvature vector field, respectively. By $\nabla^\perp$ and $\Delta^\perp$ we shall denote the normal connection and Laplacian on the normal bundle of M in N.

We have the following characterization.

Theorem 3.27 ([69]). *The Hopf cylinder $\mathbf{i} : M^m = \pi^{-1}(\bar{M}) \to N$ is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H = -\operatorname{trace} B(\cdot, A_H \cdot) + \frac{c(m+2)+3m-2}{4} H + \frac{3(c-1)}{4} (\varphi(\varphi H)^\perp)^\perp \\ \\ 4 \operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + m \operatorname{grad}(|H|^2) - 3(c-1)(\varphi(\varphi H)^\perp)^\top = 0. \end{cases} \tag{3.51}$$

Corollary 3.28 ([69]). *If $\bar{M}$ is a hypersurface of $\bar{N}$, then $M = \pi^{-1}(\bar{M})$ is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H = \Big(-|B|^2 + \frac{c(n+1)+3n-1}{2} \Big) H \\ \\ 2 \operatorname{trace} A_{\nabla^\perp_\cdot H}(\cdot) + n \operatorname{grad}(|H|^2) = 0. \end{cases} \tag{3.52}$$

Proof. This result follows easily since, in codimension 1, $(\varphi H)^\perp = 0$ and

$$\operatorname{trace} B(\cdot, A_H \cdot) = |B|^2 H.$$

□

Now, since $\tau(\mathbf{i}) = 2nH = (\tau(\bar{\mathbf{i}}))^H = (2n-1)\bar{H}^H$, we obtain the following.

Corollary 3.29 ([69]). *If $\bar{M}$ is a hypersurface and $|\bar{H}| =$ constant $\neq 0$, then $M = \pi^{-1}(\bar{M})$ is proper-biharmonic if and only if*

$$|B|^2 = \frac{c(n+1)+3n-1}{2}.$$

We shall prove now a Lawson type formula which relates $|B|^2$ to $|\bar{B}|^2$ (see [63], [76] and [86]). First we denote by $\varphi_M^\perp$ the restriction of φ to the normal bundle of M in N composed with the projection on the same normal bundle, that is $\varphi_M^\perp \sigma = (\varphi\sigma)^\perp$, for any σ a section in the normal bundle of M in N.

Proposition 3.30 ([69]). *Let $\bar{M}$ be a submanifold of $\bar{N}$, and denote by $\bar{B}$ its second fundamental form. Then, the second fundamental form B of $\pi^{-1}(\bar{M})$ in N and $\bar{B}$ are related by*

$$|B|^2 = |\bar{B}|^2 + 2(2n+1-m) - 2|\varphi_M^\perp|^2.$$

Proof. Let us consider $\bar{X}, \bar{Y} \in C(T\bar{M})$. We have

$$\begin{cases} \nabla^N_{\bar{X}^H} \bar{Y}^H = (\nabla^{\bar{N}}_{\bar{X}} \bar{Y})^H + \frac{1}{2} V[\bar{X}^H, \bar{Y}^H] \\ \\ \nabla_{\bar{X}^H} \bar{Y}^H = (\nabla^{\bar{M}}_{\bar{X}} \bar{Y})^H + \frac{1}{2} V[\bar{X}^H, \bar{Y}^H] \end{cases},$$

thus $B(\bar{X}^H, \bar{Y}^H) = (\bar{B}(\bar{X}, \bar{Y}))^H$.

Also,

$$\begin{aligned} B(\bar{X}^H, \xi) &= \textstyle\sum_{a=m+1}^{2n+1} g(B(\bar{X}^H, \xi), \sigma_a)\sigma_a = \sum_{a=m+1}^{2n+1} g(\nabla^N_{\bar{X}^H}\xi - \nabla_{\bar{X}^H}\xi, \sigma_a)\sigma_a \\ &= -\textstyle\sum_{a=m+1}^{2n+1} g(\varphi\bar{X}^H, \sigma_a)\sigma_a = \sum_{a=m+1}^{2n+1} g(\varphi\sigma_a, \bar{X}^H)\sigma_a, \end{aligned}$$

where $\{\sigma_a\}_{a=m+1}^{2n+1}$ is a local orthonormal frame in the normal bundle of M in N.

Next, let $\{\bar{X}_\alpha\}_{\alpha=1}^{m-1}$ be a local orthonormal frame on $\bar{M}$. It follows that $\{\bar{X}_\alpha^H\}_{\alpha=1}^{m-1} \cup \{\xi\}$ is a local orthonormal frame on M and one obtains

$$\begin{aligned} |B|^2 &= |B(\xi,\xi)|^2 + 2\textstyle\sum_{\alpha=1}^{m-1} |B(\bar{X}_\alpha^H, \xi)|^2 + \sum_{\alpha,\beta=1}^{m-1} |B(\bar{X}_\alpha^H, \bar{X}_\beta^H)|^2 \\ &= 2\textstyle\sum_{\alpha=1}^{m-1} \sum_{a=m+1}^{2n+1} (g(\varphi\sigma_a, \bar{X}_\alpha^H))^2 + |\bar{B}|^2 \\ &= |\bar{B}|^2 + 2(2n+1-m-\textstyle\sum_{a=m+1}^{2n+1} |(\varphi\sigma_a)^\perp|^2). \end{aligned}$$

□

Corollary 3.31 ([69]). *If $\bar{M}$ is a hypersurface, then $|B|^2 = |\bar{B}|^2 + 2$.*

From Corollary 3.29 and Corollary 3.31 we obtain the following result.

Proposition 3.32 ([69]). *If $|\bar{H}| =$ constant $\neq 0$, then $M = \pi^{-1}(\bar{M})$ is proper-biharmonic if and only if*

$$|\bar{B}|^2 = \frac{c(n+1) + 3n - 5}{2}.$$

Remark 3.33 ([69]). From Proposition 3.32 we see that there exist no proper-biharmonic hypersurfaces $M = \pi^{-1}(\bar{M})$ in $N(c)$ if $c \leq \frac{5-3n}{n+1}$, which implies that such hypersurfaces do not exist if $c \leq -3$, whatever the dimension of N is.

Proposition 3.34 ([69]). *If $M = \pi^{-1}(\bar{M})$ is a proper-biharmonic hypersurface with constant mean curvature, then*

$$|H|^2 \in \Big(0, \frac{(2n-1)(c(n+1)+3n-5)}{8n^2}\Big).$$

Proof. Assume that $M = \pi^{-1}(\bar{M})$ is a proper-biharmonic hypersurface with constant mean curvature. Then, from Corollary 3.29 and Proposition 3.32 follows that

$$|B|^2 = \frac{c(n+1)+3n-1}{2}, \quad |\bar{B}|^2 = \frac{c(n+1)+3n-5}{2}.$$

On the other hand we have the inequalities

$$|B|^2 \geq 2n|H|^2 \quad \text{and} \quad |\bar{B}|^2 \geq (2n-1)|\bar{H}|^2.$$

It can be easily proved that there are no non-minimal umbilical hypersurfaces of type $M = \pi^{-1}(\bar{M})$ and it is known that $\bar{M}$ cannot be umbilical. Therefore, in the above inequalities we cannot have equality, so

$$\frac{c(n+1)+3n-1}{2} > 2n|H|^2 \quad \text{and} \quad \frac{c(n+1)+3n-5}{2} > (2n-1)|\bar{H}|^2.$$

But

$$|H|^2 = \frac{(2n-1)^2}{(2n)^2}|\bar{H}|^2 < \frac{(2n-1)(c(n+1)+3n-5)}{8n^2}.$$

Since $\frac{(2n-1)(c(n+1)+3n-5)}{8n^2} < \frac{c(n+1)+3n-1}{4n}$, one obtains

$$|H|^2 \in \Big(0, \frac{(2n-1)(c(n+1)+3n-5)}{8n^2}\Big).$$

□

Proposition 3.35 ([69]). *If $M = \pi^{-1}(\bar{M})$ is a proper-biharmonic hypersurface with constant mean curvature, then the scalar curvature s of M is constant*

$$s = (c+3)(n^2-n) + \frac{c-1}{2}(n-3) + 4n^2|H|^2.$$

Proof. Let $\{X_i\}_{i=1}^{2n}$ be a local orthonormal frame on M.
Using the Gauss equation we have

$$\begin{aligned} g(R(X_i,X)Y,X_i) = \quad & g(R^N(X_i,X)Y,X_i) - g(B(X,X_i),B(X_i,Y)) \\ & + g(B(X_i,X_i),B(X,Y)). \end{aligned} \tag{3.53}$$

We consider $H = |H|\sigma$ and $A = A_\sigma$, where σ is a unit section in the normal bundle of M in N. We obtain

$$\begin{aligned} & \textstyle\sum_{i=1}^{2n}(g(B(X_i,X_i),B(X,Y)) - g(B(X,X_i),B(X_i,Y))) \\ & = g(2nH,B(X,Y)) - \textstyle\sum_{i=1}^{2n} g(B(X,X_i),\sigma)g(B(X_i,Y),\sigma) \\ & = 2n|H|g(\sigma,B(X,Y)) - \textstyle\sum_{i=1}^{2n} g(A(X),X_i)g(A(Y),X_i). \end{aligned} \tag{3.54}$$

In order to compute $\sum_{i=1}^{2n} g(R^N(X_i, X)Y, X_i)$ we shall use the local orthonormal frame $\{\bar{X}_\alpha^H\}_{\alpha=1}^{2n-1} \cup \{\xi\}$ on M where $\{\bar{X}_\alpha\}_{\alpha=1}^{2n-1}$ is a local orthonormal frame on $\bar{M}$. From the expression of the curvature tensor field of N we have

$$\begin{aligned} g(R^N(\bar{X}_\alpha^H, X)Y, \bar{X}_\alpha^H) = &\ \tfrac{c+3}{4}(g(X,Y) - g(X, \bar{X}_\alpha^H)g(Y, \bar{X}_\alpha^H)) \\ &+ \tfrac{c-1}{4}(-\eta(X)\eta(Y) + 3g(\varphi X, \bar{X}_\alpha^H)g(\varphi Y, \bar{X}_\alpha^H)) \end{aligned}$$

and

$$g(R^N(\xi, X)Y, \xi) = \frac{c+3}{4}(g(X,Y) - \eta(X)\eta(Y)) - \frac{c-1}{4}g(\varphi X, \varphi Y).$$

In conclusion

$$\begin{aligned} \sum_{i=1}^{2n} g(R^N(X_i, X)Y, X_i) = &\ \tfrac{(2n-1)(c+3)}{4}g(X,Y) - \tfrac{(2n-1)(c-1)}{4}\eta(X)\eta(Y) \\ &+ \tfrac{c-1}{2}g(\varphi X, \varphi Y) - \tfrac{3(c-1)}{4}g(X, \varphi\sigma)g(Y, \varphi\sigma). \end{aligned} \tag{3.55}$$

We obtain, using (3.53), (3.54), (3.55) and Corollary 3.29, the scalar curvature of M

$$\begin{aligned} s = \operatorname{trace}\operatorname{Ricci} &= \tfrac{c+3}{4}2n(2n-1) + \tfrac{c-1}{4}(2n-4) + 4n^2|H|^2 - |B|^2 \\ &= (c+3)(n^2 - n) + \tfrac{c-1}{2}(n-3) + 4n^2|H|^2. \end{aligned}$$

□

3.2.3 Classification results for biharmonic hypersurfaces in Sasakian space forms with φ-sectional curvature $c > -3$

In [123] all homogeneous real hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n > 1$, are classified and five types of such hypersurfaces are identified (see also [104]). We shall use them for classifying the proper-biharmonic Hopf cylinders $M = \pi^{-1}(\bar{M})$ in Sasakian space forms $N^{2n+1}(c)$, $c + 3 > 0$.

3.2.4 Types $A1$, $A2$

We shall consider $u \in (0, \frac{\pi}{2})$ and r a positive constant given by $\frac{1}{r^2} = \frac{c+3}{4}$. A hypersurface of Type $A1$ in $\mathbb{C}P^n(c+3)$ is a geodesic sphere and it has two distinct principal curvatures: $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2n-2$ and $a = \frac{2}{r}\cot 2u$ of multiplicity 1, while a hypersurface of Type $A2$ has three distinct principal curvatures: $\lambda_1 = -\frac{1}{r}\tan u$ of multiplicity $2p$, $\lambda_2 = \frac{1}{r}\cot u$ of multiplicity $2q$, and $a = \frac{2}{r}\cot 2u$ of multiplicity 1, where $p > 0$, $q > 0$, and $p + q = n - 1$.

We note that if $c = 1$ and $\bar{M}$ is a hypersurface of Type $A1$ or $A2$, then $\pi^{-1}(\bar{M})$ is the standard (extrinsic) product of a circle of radius $\cos u$ and a $(2n-1)$-dimensional sphere of radius $\sin u$ or, respectively, the standard product of two spheres of dimensions $2p+1$ and $2q+1$ and of radii $\cos u$ and $\sin u$.

Now, for the biharmonicity of the hypersurfaces $M = \pi^{-1}(\bar{M})$ in $N^{2n+1}(c)$, where $\bar{M}$ is a hypersurface in $\mathbb{C}P^n(c+3)$ of Type $A1$ or $A2$, we can state the following result.

Theorem 3.36 ([69])**.** *Let $M = \pi^{-1}(\bar{M})$ be the Hopf cylinder over $\bar{M}$.*

1. *If $\bar{M}$ is of Type $A1$, then M is proper-biharmonic if and only if either*
   (a) $c = 1$ *and* $(\tan u)^2 = 1$, *or*
   (b) $c \in \left[\frac{-3n^2+2n+1+8\sqrt{2n-1}}{n^2+2n+5}, +\infty\right) \setminus \{1\}$ *and*

$$(\tan u)^2 = n + \frac{2c - 2 \pm \sqrt{c^2(n^2+2n+5) + 2c(3n^2 - 2n - 1) + 9n^2 - 30n + 13}}{c+3}.$$

2. *If $\bar{M}$ is of Type $A2$, then M is proper-biharmonic if and only if either*
   (a) $c = 1$, $(\tan u)^2 = 1$ *and* $p \neq q$, *or*
   (b) $c \in \left[\frac{-3(p-q)^2-4n+4+8\sqrt{(2p+1)(2q+1)}}{(p-q)^2+4n+4}, +\infty\right) \setminus \{1\}$ *and*

$$\begin{aligned}(\tan u)^2 = \ & \tfrac{n}{2p+1} + \tfrac{2c-2}{(c+3)(2p+1)} \\ & \pm \tfrac{\sqrt{c^2((p-q)^2+4n+4)+2c(3(p-q)^2+4n-4)+9(p-q)^2-12n+4}}{(c+3)(2p+1)}.\end{aligned}$$

Proof. First, assume that $\bar{M}$ is of Type $A1$. Then, from Proposition 3.32, we have that $M = \pi^{-1}(\bar{M})$ is biharmonic if and only if

$$\begin{aligned}|\bar{B}|^2 &= (2n-2)\lambda_2^2 + a^2 = (2n-2)\tfrac{1}{r^2}(\cot u)^2 + \tfrac{4}{r^2}(\cot 2u)^2 \\ &= \tfrac{c(n+1)+3n-5}{2}.\end{aligned}$$

Denoting $\tan u = t$, after a straightforward computation, we obtain the equation

$$(c+3)t^4 - 2(c(n+2) + 3n - 2)t^2 + (2n-1)(c+3) = 0, \tag{3.56}$$

which admits real solutions if and only if

$$c^2(n^2+2n+5) + 2c(3n^2 - 2n - 1) + 9n^2 - 30n + 13 \geq 0.$$

But $c > \frac{5-3n}{n+1}$ and we can conclude that (3.56) has real solutions if and only if

$$c \in \left[\frac{-3n^2+2n+1+8\sqrt{2n-1}}{n^2+2n+5}, +\infty\right),$$

and these solutions are given by

$$t_{1,2}^2 = n + \frac{2c - 2 \pm \sqrt{c^2(n^2+2n+5) + 2c(3n^2-2n-1) + 9n^2 - 30n + 13}}{c+3} > 0.$$

Now, we have that M is minimal if and only if $\bar{M}$ is minimal and this means

$$(2n-2)\lambda_2 + a = 0,$$

which leads to $(\tan u)^2 = 2n - 1$.

It is easy to obtain that if one of the solutions t_1^2, t_2^2 is equal to $2n-1$ then $c=1$. If $c=1$, then M is proper-biharmonic if and only if $(\tan u)^2=1$, and if $c\neq 1$, then $t_1^2\neq 2n-1$ and $t_2^2\neq 2n-1$.

Next, let $\bar{M}$ be a hypersurface of Type $A2$. Then, according to Proposition 3.32, M is biharmonic if and only if

$$\begin{aligned}|\bar{B}|^2 &= 2p\lambda_1^2+2q\lambda_2^2+a^2=2p\tfrac{1}{r^2}(\tan u)^2+2q\tfrac{1}{r^2}(\cot u)^2+\tfrac{4}{r^2}(\cot 2u)^2\\ &= \tfrac{c(n+1)+3n-5}{2}.\end{aligned}$$

This equation becomes, after a straightforward computation,

$$(c+3)(2p+1)t^4-2(c(n+2)+3n-2)t^2+(c+3)(2q+1)=0, \tag{3.57}$$

where $t=\tan u$.
The equation (3.57) has real solutions if and only if

$$c^2((p-q)^2+4n+4)+2c(3(p-q)^2+4n-4)+9(p-q)^2-12n+4\geq 0,$$

which, together with $c>\frac{5-3n}{n+1}$, leads to

$$c\in\Big[\frac{-3(p-q)^2-4n+4+8\sqrt{(2p+1)(2q+1)}}{(p-q)^2+4n+4},+\infty\Big)\setminus\{1\}.$$

Then the solutions of equation (3.57) are

$$\begin{aligned}t_{1,2}^2= &\ \tfrac{n}{2p+1}+\tfrac{2c-2}{(c+3)(2p+1)}\\ &\pm\tfrac{\sqrt{c^2((p-q)^2+4n+4)+2c(3(p-q)^2+4n-4)+9(p-q)^2-12n+4}}{(c+3)(2p+1)}>0.\end{aligned} \tag{3.58}$$

The hypersurface $\bar{M}$ is minimal if and only if

$$2p\lambda_1+2q\lambda_2+a=0,$$

which gives $(\tan u)^2=\frac{2q+1}{2p+1}$. It follows that M is proper-biharmonic if $c=1$, $(\tan u)^2=1$ and $p\neq q$, or $c\neq 1$ and $\tan u$ is given by (3.58). □

Remark 3.37 ([69])**.** If $c\neq 1$, in the $A1$ case we can obtain two proper-biharmonic Hopf cylinders, not only one. The same thing happens in the $A2$ case when $p\neq q$; for $p=q$ we do obtain a proper-biharmonic Hopf cylinder if and only if $c\in(1,+\infty)$ and, in this case, it is given by:

$$(\tan u)^2=1+\frac{2(c-1)+2\sqrt{c^2(n+1)+2c(n-1)-3n+1}}{n(c+3)}.$$

3.2.5 Types B, C, D and E

We shall consider $u \in (0, \frac{\pi}{4})$ and r a positive constant given by $\frac{1}{r^2} = \frac{c+3}{4}$. The Type B hypersurfaces in complex projective space $\mathbb{C}P^n(c+3)$ have three distinct principal curvatures: $-\frac{1}{r}\cot u$ and $\frac{1}{r}\tan u$, both of multiplicity $n-1$, and $\frac{2}{r}\tan 2u$ of multiplicity 1. The hypersurfaces of Type C, D or E have five distinct principal curvatures: $\lambda_1 = -\frac{1}{r}\cot u$, $\lambda_2 = \frac{1}{r}\cot\left(\frac{\pi}{4} - u\right)$, $\lambda_3 = \frac{1}{r}\cot\left(\frac{\pi}{2} - u\right)$, $\lambda_4 = \frac{1}{r}\cot\left(\frac{3\pi}{4} - u\right)$ and $a = -\frac{2}{r}\cot 2u$, each with specific multiplicities (see [104] and [123]).

For what concerns the biharmonicity of Hopf cylinders $M = \pi^{-1}(\bar{M})$ we have the following non-existence result.

Theorem 3.38 ([69]). *There are no proper-biharmonic hypersurfaces $M = \pi^{-1}(\bar{M})$, where $\bar{M}$ is a hypersurface of Type B, C, D or E in complex projective space $\mathbb{C}P^n(c+3)$.*

Proof. First, let $\bar{M}$ be a hypersurface of Type B. Then, from Proposition 3.32 we have that M is biharmonic if and only if

$$|\bar{B}|^2 = (n-1)\tfrac{1}{r^2}((\cot u)^2 + (\tan u)^2) + \tfrac{4}{r^2}(\tan 2u)^2 = \tfrac{c(n+1)+3n-5}{2}.$$

If we denote $(\sin 2u)^2 = t$ we obtain easily the following equation

$$(cn + c + 3n - 1)t^2 - (2cn - c + 6n - 7)t + (n-1)(c+3) = 0. \tag{3.59}$$

If $c = 1$ the equation become $nt^2 - 2(n-1)t + n - 1 = 0$ and it has no real solutions. Assume that $c \neq 1$. Then equation (3.59) has real solutions if and only if

$$c^2(5 - 4n) - 2c(12n - 11) + 37 - 36n \geq 0.$$

Further, it follows that

$$c \in \Big[\frac{11 - 12n - 8\sqrt{n-1}}{4n-5}, \frac{11 - 12n + 8\sqrt{n-1}}{4n-5}\Big].$$

But since $c > \frac{5-3n}{n+1}$ it results that there are no real solutions of (3.59) if $n \leq 17$. Now, if $n > 17$ we have two real solutions

$$t_{1,2} = \frac{2cn - c + 6n - 7 \pm \sqrt{c^2(5-4n) - 2c(12n-11) + 37 - 36n}}{2(cn + c + 3n - 1)}.$$

Finally, it can be easily verified that $t_{1,2} > 1$ and this is a contradiction since $t = (\sin 2u)^2$.

Let $\bar{M}$ be a hypersurface of Type C. These hypersurfaces occur for $n \geq 5$ and n odd. The multiplicities of the principal curvatures are: $n-3$ for λ_1 and λ_3, 2 for λ_2 and λ_4, and 1 for a.
In the same way as above, by denoting $t = (\sin 2u)^2$, we have that M is biharmonic if and only if

$$(cn + c + 3n - 1)t^2 - (2cn - 3c + 6n - 13)t + (n-2)(c+3) = 0. \tag{3.60}$$

If $c = 1$ it is easy to see that (3.60) does not admit real solution. If $c \neq 1$ equation (3.60) has real solutions if and only if

$$c^2(17 - 8n) - 2c(24n - 47) + 145 - 72n \geq 0$$

and this means

$$c \in \Big[\frac{24n - 47 + 8\sqrt{2(n-2)}}{17 - 8n}, \frac{24n - 47 - 8\sqrt{2(n-2)}}{17 - 8n}\Big].$$

Since $c > \frac{5-3n}{n+1}$ it follows that real solutions exist only if $n \geq 33$ and they are

$$t_{1,2} = \frac{2cn - 3c + 6n - 13 \pm \sqrt{c^2(17 - 8n) - 2c(24n - 47) + 145 - 72n}}{2(cn + c + 3n - 1)}.$$

But $t_{1,2}$ are greater than 1 and, since $t = (\sin 2u)^2$, M cannot be proper-biharmonic.

The hypersurfaces of Type D occurs only in $\mathbb{C}P^9(c+3)$. In this case, the multiplicity of each of the first four principal curvatures is 4 and the multiplicity of the fifth one is 1.

Now, let $\bar{M}$ be a hypersurface of Type D. As in the previous two cases, we obtain that M is biharmonic if and only if

$$(10c + 26)t^2 - (11c + 29)t + 5c + 15 = 0,$$

where $t = (\sin 2u)^2$. Real solutions exist if and only if

$$c \in \Big[-\frac{241 + 16\sqrt{5}}{79}, -\frac{241 - 16\sqrt{5}}{79}\Big].$$

But $c > \frac{5-3n}{n+1} = -\frac{11}{5}$ and $-\frac{11}{5} > -\frac{241-16\sqrt{5}}{79}$. Thus there are no real solutions.

Finally, let $\bar{M}$ be a hypersurface of Type E. This case occurs only in $\mathbb{C}P^{15}(c+3)$, and the multiplicities are: 8 for λ_1 and λ_3, 6 for λ_2 and λ_4 and 1 for the principal curvature a.

It follows that M is biharmonic if and only if

$$(16c + 44)t^2 - (19c + 53)t + 9c + 27 = 0,$$

where $t = (\sin 2u)^2$. The equation has real solutions if and only if

$$c \in \Big[-\frac{649 + 24\sqrt{6}}{215}, -\frac{649 - 24\sqrt{6}}{215}\Big].$$

Since $c > \frac{5-3n}{n+1} = -\frac{5}{2}$ and $-\frac{5}{2} > -\frac{649-24\sqrt{6}}{215}$, there exist no real solutions. $\square$

3.3 Biharmonic integral $\mathcal{C}$-parallel submanifolds in 7-dimensional Sasakian space forms

3.3.1 Introduction

We start the last section of Chapter 3 by recalling some general facts on Sasakian space forms with a special emphasis on the notion of integral $\mathcal{C}$-parallel submanifolds. Then we study the biharmonicity of maximum dimensional integral submanifolds in a Sasakian space form. We obtain the necessary and sufficient conditions for such a submanifold to be biharmonic, we prove some non-existence results and we find the characterization of proper-biharmonic integral $\mathcal{C}$-parallel submanifolds of maximum dimension. Restricting our attention on 7-dimensional Sasakian space forms, we classify all 3-dimensional proper-biharmonic integral $\mathcal{C}$-parallel submanifolds in a 7-dimensional Sasakian space form, and we find these submanifolds in the 7-sphere endowed with its canonical and deformed Sasakian structures introduced by S. Tanno in [125]. A key ingredient proved to be a special local basis constructed on the 3-dimensional integral $\mathcal{C}$-parallel submanifolds.

In the last part we classify the proper-biharmonic parallel Lagrangian submanifolds of $\mathbb{C}P^3$ by determining their horizontal lifts, with respect to the Hopf fibration, in $\mathbb{S}^7(1)$.

3.3.2 Integral $\mathcal{C}$-parallel submanifolds of a Sasakian manifold

We recall that a submanifold M^m of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ is called an *integral submanifold* if $\eta(X) = 0$ for any vector field X tangent to M. We have $\varphi(TM) \subset NM$ and $m \leq n$, where TM and NM are the tangent bundle and the normal bundle of M, respectively. Moreover, for $m = n$, one gets $\varphi(NM) = TM$. If we denote by B the second fundamental form of M then, by a straightforward computation, one obtains the relation

$$g(\varphi Z, B(X, Y)) = g(\varphi Y, B(X, Z))$$

for any vector fields X, Y and Z tangent to M (see also [10]). We also note that $A_\xi = 0$, where A is the shape operator of M (see [26]).

A submanifold $\widetilde{M}$ of N is said to be *anti-invariant* if ξ is tangent to $\widetilde{M}$ and φ maps the tangent bundle to $\widetilde{M}$ into its normal bundle.

Next, we shall recall the notion of an integral $\mathcal{C}$-parallel submanifold of a Sasakian manifold (see, for example, [10]). Let M^m be an integral submanifold of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$. Then M is said to be *integral $\mathcal{C}$-parallel* if $\nabla^\perp B$ is parallel to the characteristic vector field ξ, where $\nabla^\perp B$ is given by

$$(\nabla^\perp B)(X, Y, Z) = \nabla^\perp_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields X, Y, Z tangent to M, $\nabla^\perp$ and ∇ being the normal connection and the Levi-Civita connection on M, respectively. Thus, M^m is an integral $\mathcal{C}$-parallel submanifold if $(\nabla^\perp B)(X, Y, Z) = S(X, Y, Z)\xi$ for any vector fields X, Y, Z tangent to M, where $S(X, Y, Z) = g(\varphi X, B(Y, Z))$ is a totally symmetric tensor field of type $(0, 3)$ on M. It is not difficult to check that, when $m = n$, $\nabla^\perp B = 0$ if and only if $B = 0$, i.e., M^n is totally geodesic.

Now, let M^m be an integral submanifold of a Sasakian manifold N^{2n+1}, and denote by H its mean curvature vector field. We say that H is *$\mathcal{C}$-parallel* if $\nabla^\perp H$ is parallel to ξ, i.e., $\nabla^\perp_X H = \theta(X)\xi$, where θ is a 1-form on M. As we shall see, $\theta(X) = g(H, \varphi X)$ for any vector field X tangent to M.

In general, a Riemannian submanifold M of N is called *parallel* if $\nabla^\perp B = 0$, and we say that H is *parallel* if $\nabla^\perp H = 0$.

The following two results shall be used later in this paper and, for the sake of completeness, we also provide their proofs.

Proposition 3.39 ([67]). *If the mean curvature vector field H of an integral submanifold M^n of a Sasakian manifold $(N^{2n+1}, \varphi, \xi, \eta, g)$ is parallel then M^n is minimal.*

Proof. Let X, Y be two vector fields tangent to M. Since

$$g(B(X,Y),\xi) = g(\nabla^N_X Y,\xi) = -g(Y,\nabla^N_X \xi) = g(Y,\varphi X) = 0$$

we have $B(X,Y) \in \varphi(TM)$ and, in particular, $H \in \varphi(TM)$. Then

$$g(\nabla^\perp_X H,\xi) = g(\nabla^N_X H,\xi) = -g(H,\nabla^N_X \xi) = g(H,\varphi X).$$

Thus, if $\nabla^\perp H = 0$ it follows that $g(H,\varphi X) = 0$ for any vector field X tangent to M, and this means $H = 0$, since M has maximal dimension. □

Proposition 3.40 ([67]). *Let $(N^{2n+1},\varphi,\xi,\eta,g)$ be a Sasakian manifold and M^m be an integral $\mathcal{C}$-parallel submanifold with mean curvature vector field H. The following hold:*

1. *$\nabla^\perp_X H = g(H,\varphi X)\xi$, for any vector field X tangent to M, i.e., H is $\mathcal{C}$-parallel;*
2. *the mean curvature $|H|$ is constant;*
3. *if $m = n$, then $\Delta^\perp H = H$.*

Proof. In order to prove (1), we consider $\{X_i\}_{i=1}^m$ to be a local geodesic frame at $p \in M$. Then we have at p

$$(\nabla^\perp B)(X_i,X_j,X_j) = \nabla^\perp_{X_i} B(X_j,X_j) = g(B(X_j,X_j),\varphi X_i)\xi$$

and, by summing for $j = 1,\ldots,m$, we obtain $\nabla^\perp_{X_i} H = g(H,\varphi X_i)\xi$. Then, for (2), we have

$$X(|H|^2) = 2g(H,\nabla^\perp_X H) = 2g(H,\varphi X)g(H,\xi) = 0$$

for any vector field X tangent to M, i.e., $|H|$ is constant.

For the last item, we assume that $m = n$. As $\nabla^N_X \xi = -\varphi X$, from the Weingarten equation, we get $A_\xi = 0$, where A_ξ is the shape operator of M corresponding to ξ, and $\nabla^\perp_X \xi = \nabla^N_X \xi = -\varphi X$. Thus

$$\begin{aligned}
\Delta^\perp H &= -\sum_{i=1}^n \nabla^\perp_{X_i}\nabla^\perp_{X_i} H = -\sum_{i=1}^n \nabla^\perp_{X_i}(g(H,\varphi X_i)\xi) \\
&= -\sum_{i=1}^n X_i(g(H,\varphi X_i))\xi - \sum_{i=1}^n g(H,\varphi X_i)\nabla^N_{X_i}\xi \\
&= -\sum_{i=1}^n X_i(g(H,\varphi X_i))\xi + \sum_{i=1}^n g(H,\varphi X_i)\varphi X_i \\
&= -\sum_{i=1}^n X_i(g(H,\varphi X_i))\xi + H.
\end{aligned}$$

But, since $\nabla^N_{X_i}\varphi X_i = \varphi\nabla^N_{X_i} X_i + \xi$, it results

$$\begin{aligned}
X_i(g(H,\varphi X_i)) &= g(\nabla^N_{X_i} H,\varphi X_i) + g(H,\varphi\nabla^N_{X_i} X_i + \xi) \\
&= g(-A_H X_i + \nabla^\perp_{X_i} H,\varphi X_i) + g(H,\varphi B(X_i,X_i)) \\
&= 0.
\end{aligned}$$

We have just proved that $\Delta^\perp H = H$. □

3.3.3 Biharmonic submanifolds in $\mathbb{S}^{2n+1}(1)$

Working with anti-invariant submanifolds rather than with cylinders, we can state the following (known) result.

Proposition 3.41 ([67]). *Let $\widetilde{M}^{m+1}$ be an anti-invariant submanifold of the strictly regular Sasakian space form $N^{2n+1}(c)$, $1 \leq m \leq n$, invariant under the flow-action of the characteristic vector field ξ. Then $\widetilde{M}$ is locally isometric to $I \times M^m$, where M^m is an integral submanifold of N. Moreover, we have*

1. *$\widetilde{M}$ is proper-biharmonic if and only if M is proper-biharmonic in N;*
2. *if $m = n$, then $\widetilde{M}$ is parallel if and only if M is $\mathcal{C}$-parallel;*
3. *if $m = n$, then the mean curvature vector field of $\widetilde{M}$ is parallel if and only if the mean curvature vector field of M is $\mathcal{C}$-parallel.*

Proof. The restriction $\xi_{/\widetilde{M}}$ of the characteristic vector field ξ to $\widetilde{M}$ is a Killing vector field tangent to $\widetilde{M}$. Since $\widetilde{M}$ is anti-invariant, the horizontal distribution defined on $\widetilde{M}$ is integrable. Let $p \in \widetilde{M}$ be an arbitrary point and M a small enough integral submanifold of the horizontal distribution on $\widetilde{M}$ such that $p \in M$. Then $F : I \times M \to F(I \times M) \subset \widetilde{M}$, $F(t,p) = \phi_t(p)$, is an isometry. As M is an integral submanifold of the horizontal distribution on $\widetilde{M}$, it is an integral submanifold of N.

The item (1) follows immediately from Theorem 3.22, and (2) and (3) are known and can be checked by straightforward computations. $\square$

We recall that, if $\widetilde{M}^2$ is a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ, then it is also anti-invariant and, locally, $\widetilde{M}$ is given by $F(t,s) = \phi_t(\gamma(s))$, where γ is a Legendre curve in N. Moreover, $\widetilde{M}$ is proper-biharmonic if and only if γ is proper-biharmonic in N.

Now, consider $\widetilde{M}^2$ a surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ and let $T = \gamma'$ and E_2 be the first two vector fields defined by the Frenet equations of the above Legendre curve γ. As $\nabla^F_{\partial/\partial t}\tau(F) = -\varphi(\tau(F))$, where ∇^F is the pull-back connection determined by the Levi-Civita connection on N, we can prove the following proposition.

Proposition 3.42 ([67]). *Let $\widetilde{M}^2$ be a proper-biharmonic surface of $N^{2n+1}(c)$ invariant under the flow-action of the characteristic vector field ξ. Then $\widetilde{M}$ has parallel mean curvature vector field if and only if $c > 1$ and $\varphi T = \pm E_2$.*

Corollary 3.43 ([67]). *The proper-biharmonic surfaces of $\mathbb{S}^{2n+1}(1)$ invariant under the flow-action of the characteristic vector field ξ_0 are not of parallel mean curvature vector field.*

We shall see that we do have examples of maximum dimensional proper-biharmonic anti-invariant submanifolds of $\mathbb{S}^{2n+1}(1)$, invariant under the flow-action of ξ_0, which have parallel mean curvature vector field.

In [120] the parametric equations of all proper-biharmonic integral surfaces in $\mathbb{S}^5(1)$ were obtained. Up to an isometry of $\mathbb{S}^5(1)$ which preserves ξ_0, we have only one proper-biharmonic integral surface given by

$$x(u,v) = \frac{1}{\sqrt{2}}(\exp(\mathrm{i}u), \mathrm{i}\exp(-\mathrm{i}u)\sin(\sqrt{2}v), \mathrm{i}\exp(-\mathrm{i}u)\cos(\sqrt{2}v)).$$

The map x induces a proper-biharmonic Riemannian embedding from the 2-dimensional torus $\mathcal{T}^2 = \mathbb{R}^2/\Lambda$ into $\mathbb{S}^5(1)$, where Λ is the lattice generated by the vectors $(2\pi, 0)$ and $(0, \sqrt{2}\pi)$.

Remark 3.44 ([67])**.** We recall that an isometric immersion $x : M \to \mathbb{R}^{n+1}$ of a compact manifold is said to be of *k-type* if its spectral decomposition contains exactly k non-constant terms excepting the center of mass $x_0 = (\mathrm{Vol}(M))^{-1}\int_M x\ v_g$. When $x_0 = 0$, the submanifold is called *mass-symmetric* (see [42]). It was proved in [18, 21] that a proper-biharmonic compact constant mean curvature submanifold M^m of $\mathbb{S}^n$ is either a 1-type submanifold of $\mathbb{R}^{n+1}$ with center of mass of norm equal to $1/\sqrt{2}$, or a mass-symmetric 2-type submanifold of $\mathbb{R}^{n+1}$. Now, using [8, Theorem 3.5], where all mass-symmetric 2-type integral surfaces in $\mathbb{S}^5(1)$ were determined, and [29, Proposition 4.1], the result in [120] can be (partially) reobtained.

Further, we consider the cylinder over x and we recover the result in [4]: up to an isometry of $\mathbb{S}^5(1)$ which preserves ξ_0, we have only one 3-dimensional proper-biharmonic anti-invariant submanifold of $\mathbb{S}^5(1)$ invariant under the flow-action of ξ_0,

$$F(t,u,v) = \exp(-\mathrm{i}t)x(u,v).$$

The map y is a proper-biharmonic Riemannian immersion with parallel mean curvature vector field and it induces a proper-biharmonic Riemannian immersion from the 3-dimensional torus $\mathcal{T}^3 = \mathbb{R}^3/\Lambda$ into $\mathbb{S}^5$, where Λ is the lattice generated by the vectors $(2\pi, 0, 0)$, $(0, 2\pi, 0)$ and $(0, 0, \sqrt{2}\pi)$. Moreover, a closer look shows that y factorizes to a proper-biharmonic Riemannian embedding in $\mathbb{S}^5$, and its image is the standard (extrinsic) product between three Euclidean circles, one of radius $1/\sqrt{2}$ and each of the other two of radius $1/2$. Indeed, we may consider the orthogonal transformation of $\mathbb{R}^3$ given by

$$T(t,u,v) = \Big(\frac{-t+u}{\sqrt{2}}, \frac{-t-u}{\sqrt{2}}, v\Big) = (t', u', v')$$

and the map y becomes

$$F_1(t',u',v') = \frac{1}{\sqrt{2}}(\exp(\mathrm{i}\sqrt{2}t'), \mathrm{i}\exp(\mathrm{i}\sqrt{2}u')\sin(\sqrt{2}v'), \mathrm{i}\exp(\mathrm{i}\sqrt{2}u')\cos(\sqrt{2}v')).$$

Then, acting with an appropriate holomorphic isometry of $\mathbb{C}^4$, y_1 becomes

$$F_2(t',u',v') = \Big(\frac{1}{\sqrt{2}}\exp(\mathrm{i}\sqrt{2}t'), \frac{1}{2}\exp(\mathrm{i}(u'-v')), \frac{1}{2}\exp(\mathrm{i}(u'+v'))\Big)$$

and, further, an obvious orthogonal transformation of the domain leads to the desired results.

3.3.4 Biharmonic integral submanifolds of maximum dimension in Sasakian space forms

Let $(N^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian space form with constant φ-sectional curvature c, and M^n an n-dimensional integral submanifold of N. We recall that this means $\eta(X) = 0$ for any vector field X tangent to M. We shall denote by B, A and H the second fundamental form of M in N, the shape operator and the mean curvature vector field, respectively. By $\nabla^\perp$ and $\Delta^\perp$ we shall denote the connection and the Laplacian in the normal bundle. We have the following theorem.

Theorem 3.45 ([67])**.** *The integral submanifold* $\mathbf{i} : M^n \to N^{2n+1}$ *is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H + \operatorname{trace} B(\cdot, A_H \cdot) - \dfrac{c(n+3)+3n-3}{4} H = 0, \\ 4 \operatorname{trace} A_{\nabla^\perp_{(\cdot)} H}(\cdot) + n \operatorname{grad}(|H|^2) = 0. \end{cases} \tag{3.61}$$

Corollary 3.46 ([67])**.** *Let* $N^{2n+1}(c)$ *be a Sasakian space form with constant* φ*-sectional curvature* $c \leq (3-3n)/(n+3)$*. Then an integral submanifold* M^n *with constant mean curvature* $|H|$ *in* $N^{2n+1}(c)$ *is biharmonic if and only if it is minimal.*

Proof. Assume that M^n is a biharmonic integral submanifold with constant mean curvature $|H|$ in $N^{2n+1}(c)$. It follows, from Theorem 3.45, that

$$\begin{aligned} g(\Delta^\perp H, H) &= -g(\operatorname{trace} B(\cdot, A_H \cdot), H) + \frac{c(n+3)+3n-3}{4}|H|^2 \\ &= \frac{c(n+3)+3n-3}{4}|H|^2 - \sum_{i=1}^n g(B(X_i, A_H X_i), H) \\ &= \frac{c(n+3)+3n-3}{4}|H|^2 - \sum_{i=1}^n g(A_H X_i, A_H X_i) \\ &= \frac{c(n+3)+3n-3}{4}|H|^2 - |A_H|^2. \end{aligned}$$

Thus, from the Weitzenböck formula

$$\frac{1}{2}\Delta|H|^2 = g(\Delta^\perp H, H) - |\nabla^\perp H|^2,$$

one obtains

$$\frac{c(n+3)+3n-3}{4}|H|^2 - |A_H|^2 - |\nabla^\perp H|^2 = 0. \tag{3.62}$$

If $c < (3-3n)/(n+3)$, relation (3.62) is equivalent to $H = 0$. Now, assume that $c = (3-3n)/(n+3)$. As for integral submanifolds $\nabla^\perp H = 0$ is equivalent to $H = 0$, again (3.62) is equivalent to $H = 0$. $\square$

Corollary 3.47 ([67])**.** *Let* $N^{2n+1}(c)$ *be a Sasakian space form with constant* φ*-sectional curvature* $c \leq (3-3n)/(n+3)$*. Then a compact integral submanifold* M^n *is biharmonic if and only if it is minimal.*

Proof. Assume that M^n is a biharmonic compact integral submanifold. As in the proof of Corollary 3.46 we have

$$g(\Delta^{\perp}H, H) = \frac{c(n+3)+3n-3}{4}|H|^2 - |A_H|^2$$

and so $\Delta|H|^2 \leq 0$, which implies that $|H|^2$ is constant. Therefore we obtain that M is minimal in this case too. □

Remark 3.48 ([67])**.** From Corollaries 3.46 and 3.47 it is easy to see that in a Sasakian space form $N^{2n+1}(c)$ with constant φ-sectional curvature $c+3 \leq 0$ a biharmonic compact integral submanifold, or a biharmonic integral submanifold M^n with constant mean curvature, is minimal whatever the dimension of N is.

Proposition 3.49 ([67])**.** *Let $N^{2n+1}(c)$ be a Sasakian space form and $\mathbf{i} : M^n \to N^{2n+1}$ be an integral $\mathcal{C}$-parallel submanifold. Then $(\tau_2(\mathbf{i}))^{\top} = 0$.*

Proof. Indeed, from Proposition 3.40 we have $|H|$ is constant and $\nabla^{\perp}H$ is parallel to ξ, which implies that $A_{\nabla^{\perp}_X H} = 0$ for any vector field X tangent to M, since $A_{\xi} = 0$. Thus we conclude the proof. □

Proposition 3.50 ([67])**.** *A non-minimal integral $\mathcal{C}$-parallel submanifold M^n of a Sasakian space form $N^{2n+1}(c)$ is proper-biharmonic if and only if $c > (7-3n)/(n+3)$ and*

$$\operatorname{trace} B(\cdot, A_H\cdot) = \frac{c(n+3)+3n-7}{4}H.$$

Proof. We know, from Proposition 3.40, that $\Delta^{\perp}H = H$. Hence, from Theorem 3.45 and the above Proposition, it follows that M^n is biharmonic if and only if

$$\operatorname{trace} B(\cdot, A_H\cdot) = \frac{c(n+3)+3n-7}{4}H.$$

Next, if M^n verifies the above condition, we contract with H and get

$$|A_H|^2 = \frac{c(n+3)+3n-7}{4}|H|^2.$$

Since A_H and H do not vanish it follows that $c > (7-3n)/(n+3)$. □

Now, let $\{X_i\}_{i=1}^n$ be an arbitrary orthonormal local frame field on the integral $\mathcal{C}$-parallel submanifold M^n of a Sasakian space form $N^{2n+1}(c)$, and let $A_i = A_{\varphi X_i}$, $i = 1, \ldots, n$, be the corresponding shape operators. Then, from Proposition 3.50, we obtain

Proposition 3.51 ([67])**.** *A non-minimal integral $\mathcal{C}$-parallel submanifold M^n of a Sasakian space form $N^{2n+1}(c)$, $c > (7-3n)/(n+3)$, is proper-biharmonic if and only if*

$$\begin{pmatrix} g(A_1, A_1) & \ldots & g(A_1, A_n) \\ \vdots & \vdots & \vdots \\ g(A_n, A_1) & \ldots & g(A_n, A_n) \end{pmatrix} \begin{pmatrix} \operatorname{trace} A_1 \\ \vdots \\ \operatorname{trace} A_n \end{pmatrix} = k \begin{pmatrix} \operatorname{trace} A_1 \\ \vdots \\ \operatorname{trace} A_n \end{pmatrix}.$$

where $k = (c(n+3)+3n-7)/4$.

3.3.5 3-dimensional biharmonic integral $\mathcal{C}$-parallel submanifolds of a Sasakian space form $N^7(c)$

In [10], C. Baikoussis, D.E. Blair and T. Koufogiorgios classified the 3-dimensional integral $\mathcal{C}$-parallel submanifolds in a Sasakian space form $(N^7(c), \varphi, \xi, \eta, g)$. In order to obtain the classification, they worked with a special local orthonormal basis (see also [57]). Here we shall briefly recall how this basis is constructed.

Let $\mathbf{i} : M^3 \to N^7(c)$ be an integral $\mathcal{C}$-parallel submanifold of constant mean curvature. Let p be an arbitrary point of M, and consider the function $f_p : U_pM \to \mathbb{R}$ given by

$$f_p(u) = g(B(u,u), \varphi u),$$

where $U_pM = \{u \in T_pM : g(u,u) = 1\}$ is the unit sphere in the tangent space T_pM. If $f_p(u) = 0$, for all $u \in U_pM$, then, for any $v_1, v_2 \in U_pM$ such that $g(v_1, v_2) = 0$ we have that

$$g(B(v_1,v_1), \varphi v_1) = 0 \quad \text{and} \quad g(B(v_1,v_1), \varphi v_2) = 0.$$

We obtain $B(v_1, v_1) = 0$, and then it follows that B vanishes at the point p.

Next, assume that the function f_p does not vanish identically. Since U_pM is compact, f_p attains an absolute maximum at a unit vector X_1. It follows that

$$\begin{cases} g(B(X_1,X_1), \varphi X_1) > 0, & g(B(X_1,X_1), \varphi X_1) \geq |g(B(w,w), \varphi w)| \\ g(B(X_1,X_1), \varphi w) = 0, & g(B(X_1,X_1), \varphi X_1) \geq 2g(B(w,w), \varphi X_1), \end{cases}$$

where w is a unit vector tangent to M at p and orthogonal to X_1. It is easy to see that X_1 is an eigenvector of the shape operator $A_1 = A_{\varphi X_1}$ with the corresponding eigenvalue λ_1. Then, since A_1 is symmetric, we consider X_2 and X_3 to be unit eigenvectors of A_1, orthogonal to each other and to X_1, with the corresponding eigenvalues λ_2 and λ_3. Further, we distinguish two cases.

If $\lambda_2 \neq \lambda_3$, we can choose X_2 and X_3 such that

$$\begin{cases} g(B(X_2,X_2), \varphi X_2) \geq 0, & g(B(X_3,X_3), \varphi X_3) \geq 0 \\ g(B(X_2,X_2), \varphi X_2) \geq g(B(X_3,X_3), \varphi X_3). \end{cases}$$

If $\lambda_2 = \lambda_3$, we consider $f_{1,p}$ the restriction of f_p to $\{w \in U_pM : g(w, X_1) = 0\}$, and we have two subcases:

1. the function $f_{1,p}$ is identically zero. In this case, we have

$$\begin{cases} g(B(X_2,X_2), \varphi X_2) = 0, & g(B(X_2,X_2), \varphi X_3) = 0 \\ g(B(X_2,X_3), \varphi X_3) = 0, & g(B(X_3,X_3), \varphi X_3) = 0. \end{cases}$$

2. the function $f_{1,p}$ does not vanish identically. Then we choose X_2 such that $f_{1,p}(X_2)$ is an absolute maximum. We have that

$$\begin{cases} g(B(X_2,X_2), \varphi X_2) > 0, & g(B(X_2,X_2), \varphi X_2) \geq g(B(X_3,X_3), \varphi X_3) \geq 0 \\ g(B(X_2,X_2), \varphi X_3) = 0, & g(B(X_2,X_2), \varphi X_2) \geq 2g(B(X_3,X_3), \varphi X_2). \end{cases}$$

Now, with respect to the orthonormal basis $\{X_1, X_2, X_3\}$, the shape operators A_1, $A_2 = A_{\varphi X_2}$ and $A_3 = A_{\varphi X_3}$, at p, can be written as follows

$$A_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & \alpha & \beta \\ 0 & \beta & \mu \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & \lambda_3 \\ 0 & \beta & \mu \\ \lambda_3 & \mu & \delta \end{pmatrix}. \tag{3.63}$$

We also have $A_0 = A_\xi = 0$. With these notations we have

$$\lambda_1 > 0, \quad \lambda_1 \geq |\alpha|, \quad \lambda_1 \geq |\delta|, \quad \lambda_1 \geq 2\lambda_2, \quad \lambda_1 \geq 2\lambda_3. \tag{3.64}$$

For $\lambda_2 \neq \lambda_3$ we get

$$\alpha \geq 0, \quad \delta \geq 0 \quad \text{and} \quad \alpha \geq \delta \tag{3.65}$$

and for $\lambda_2 = \lambda_3$ we obtain that

$$\alpha = \beta = \mu = \delta = 0 \tag{3.66}$$

or

$$\alpha > 0, \quad \delta \geq 0, \quad \alpha \geq \delta, \quad \beta = 0 \quad \text{and} \quad \alpha \geq 2\mu. \tag{3.67}$$

Now, let V_p be a normal neighbourhood around the point p. Consider q an arbitrary point in V_p, and let $\gamma_q : [0, 1] \to V_p$ be the unique geodesic with $\gamma_q(0) = p$ and $\gamma_q(1) = q$. For an arbitrary vector $u \in T_pM$ we consider its parallel transport $u(t)$ along the geodesic γ_q; $u = u(0)$. It is not difficult to check that the function

$$t \longrightarrow f_{\gamma_q(t)}(u(t)) = g(B(u(t), u(t)), \varphi(t))$$

is constant and thus $f_p(u) = f_q(u(1))$. Therefore, f_q vanishes identically if and only if f_p vanishes too, i.e. $B_q = 0$ if and only if $B_p = 0$.

Assume that f_p does not vanish identically, and consider $X_1(t)$ the parallel transport of X_1 along γ_q. The function $f_{\gamma_q(t)}$ attains an absolute maximum at $X_1(t)$. We define $A_1 = A_{\varphi X_1}$ along γ_q and we have that $X_1(t)$ is a an eigenvector of A_1. Again, we consider $X_2(t)$ and $X_3(t)$ the parallel transport of X_2 and X_3, respectively, along γ. It follows that $X_2(t)$ and $X_3(t)$ are eigenvectors for A_1 and λ_1, λ_2, λ_3 are constant allong γ_q. As the function

$$t \longrightarrow g(B(X_i, X_j), \varphi(X_k))$$

is constant, it follows that the functions α, β and μ are constant along γ_q.

Using this basis and de Rham decomposition theorem, in [10], the authors classified locally all 3-dimensional integral $\mathcal{C}$-parallel submanifolds in a 7-dimensional Sasakian space form.

According to that classification, if $c + 3 > 0$ then M is a non-minimal integral $\mathcal{C}$-parallel submanifold if and only if either:

Case I. M is flat, it is locally a product of three curves which are helices of osculating orders $r \leq 4$, and $\lambda_1 = (\lambda^2 - (c+3)/4)/\lambda$, $\lambda_2 = \lambda_3 = \lambda = \text{constant} \neq 0$, $\alpha = \text{constant}$, $\beta = 0$, $\mu = \text{constant}$, $\delta = \text{constant}$, such that $-\sqrt{c+3}/2 < \lambda < 0$, $0 < \alpha \leq \lambda_1$, $\alpha > 2\mu$, $\alpha \geq \delta \geq 0$, $(c+3)/4 + \lambda^2 + \alpha\mu - \mu^2 = 0$ and $((3\lambda^2 - (c+3)/4)/\lambda)^2 + (\alpha + \mu)^2 + \delta^2 > 0$, or

Case II. M is locally isometric to a product $\gamma \times \bar{M}^2$, where γ is a curve and $\bar{M}^2$ is a $\mathcal{C}$-parallel surface, and either

(1) $\lambda_1 = 2\lambda_2 = -\lambda_3 = \sqrt{c+3}/(2\sqrt{2})$, $\alpha = \mu = \delta = 0$, $\beta = \pm\sqrt{3(c+3)}/(4\sqrt{2})$. In this case γ is a helix in N with curvatures $\kappa_1 = 1/\sqrt{2}$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere of radius $\rho = \sqrt{8/(3(c+3))}$, or

(2) $\lambda_1 = (\lambda^2 - (c+3)/4)/\lambda$, $\lambda_2 = \lambda_3 = \lambda = \text{constant}$, $\alpha = \beta = \mu = \delta = 0$, such that $-\sqrt{c+3}/2 < \lambda < 0$ and $\lambda^2 \neq (c+3)/12$. In this case, γ is a helix in N with curvatures $\kappa_1 = \lambda_1$ and $\kappa_2 = 1$, and $\bar{M}^2$ is the 2-dimensional Euclidean sphere of radius $\rho = 1/\sqrt{(c+3)/4 + \lambda^2}$.

Now, identifying the shape operators A_i with the corresponding matrices, from Proposition 3.51, we get the following proposition.

Proposition 3.52 ([67]). *A non-minimal integral $\mathcal{C}$-parallel submanifold M^3 of a Sasakian space form $N^7(c)$, $c > -1/3$, is proper-biharmonic if and only if*

$$\left(\sum_{i=1}^{3} A_i^2\right)\begin{pmatrix} \operatorname{trace} A_1 \\ \operatorname{trace} A_2 \\ \operatorname{trace} A_3 \end{pmatrix} = \frac{3c+1}{2}\begin{pmatrix} \operatorname{trace} A_1 \\ \operatorname{trace} A_2 \\ \operatorname{trace} A_3 \end{pmatrix}, \tag{3.68}$$

where matrices A_i are given by (3.63).

Now, we can state the theorem.

Theorem 3.53 ([67]). *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of a Sasakian space form $N^7(c)$ is proper-biharmonic if and only if either:*

1. *$c > -1/3$ and M^3 is flat and it is locally a product of three curves:*
   - *a helix with curvatures $\kappa_1 = (\lambda^2 - (c+3)/4)/\lambda$ and $\kappa_2 = 1$,*
   - *a helix of order 4 with curvatures $\kappa_1 = \sqrt{\lambda^2 + \alpha^2}$, $\kappa_2 = (\alpha/\kappa_1)\sqrt{\lambda^2+1}$ and $\kappa_3 = -(\lambda/\kappa_1)\sqrt{\lambda^2+1}$,*
   - *a helix of order 4 with curvatures $\kappa_1 = \sqrt{\lambda^2+\mu^2+\delta^2}$, $\kappa_2 = (\delta/\kappa_1)\sqrt{\lambda^2+\mu^2+1}$ and $\kappa_3 = (\kappa_2/\delta)\sqrt{\lambda^2+\mu^2}$, if $\delta \neq 0$, or a circle with curvature $\kappa_1 = \sqrt{\lambda^2+\mu^2}$, if $\delta = 0$,*

   where $\lambda, \alpha, \mu, \delta$ are constants given by

$$\begin{cases} \left(3\lambda^2 - \dfrac{c+3}{4}\right)\left(3\lambda^4 - 2(c+1)\lambda^2 + \dfrac{(c+3)^2}{16}\right) + \lambda^4((\alpha+\mu)^2+\delta^2) = 0, \\ (\alpha+\mu)\left(5\lambda^2 + \alpha^2 + \mu^2 - \dfrac{7c+5}{4}\right) + \mu\delta^2 = 0, \\ \delta\left(5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - \dfrac{7c+5}{4}\right) = 0, \\ \dfrac{c+3}{4} + \lambda^2 + \alpha\mu - \mu^2 = 0 \end{cases} \tag{3.69}$$

   such that $-\sqrt{c+3}/2 < \lambda < 0$, $0 < \alpha \leq (\lambda^2 - (c+3)/4)/\lambda$, $\alpha \geq \delta \geq 0$, $\alpha > 2\mu$ and $\lambda^2 \neq (c+3)/12$;

or

(2) M^3 is locally isometric to a product $\gamma \times \bar{M}^2$ between a curve and a $\mathcal{C}$-parallel surface of N, and either

(a) $c = 5/9$, γ is a helix in $N^7(5/9)$ with curvatures $\kappa_1 = 1/\sqrt{2}$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $\sqrt{3}/2$, or

(b) $c \in [(-7 + 8\sqrt{3})/13, +\infty) \setminus \{1\}$, γ is a helix in $N^7(c)$ with curvatures $\kappa_1 = (\lambda^2 - (c+3)/4)/\lambda$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $2/\sqrt{4\lambda^2 + c + 3}$, where

$$\lambda < 0 \quad \textit{and} \quad \lambda^2 = \begin{cases} \dfrac{4c + 4 \pm \sqrt{13c^2 + 14c - 11}}{12} & \textit{if} \quad c < 1, \\ \dfrac{4c + 4 - \sqrt{13c^2 + 14c - 11}}{12} & \textit{if} \quad c > 1. \end{cases} \tag{3.70}$$

Proof. Let M^3 be a proper-biharmonic integral $\mathcal{C}$-parallel submanifold of a Sasakian space form $N^7(c)$. From Proposition 3.52 we see that $c > -1/3$.

Next, we easily get that the equation (3.68) is equivalent to the system

$$\begin{cases} \left(\sum_{i=1}^{3} \lambda_i\right)\left(\sum_{i=1}^{3} \lambda_i^2 - \dfrac{3c+1}{2}\right) + (\alpha + \mu)(\alpha\lambda_2 + \mu\lambda_3) + (\beta + \delta)(\beta\lambda_2 + \delta\lambda_3) = 0, \\ \left(\sum_{i=1}^{3} \lambda_i\right)(\alpha\lambda_2 + \mu\lambda_3) + (\alpha + \mu)\left(2\lambda_2^2 + \alpha^2 + 3\beta^2 + \mu^2 + \beta\delta - \dfrac{3c+1}{2}\right) + \mu(\beta + \delta)^2 = 0, \\ \left(\sum_{i=1}^{3} \lambda_i\right)(\beta\lambda_2 + \delta\lambda_3) + \beta(\alpha + \mu)^2 + (\beta + \delta)\left(2\lambda_3^2 + \delta^2 + 3\mu^2 + \beta^2 + \alpha\mu - \dfrac{3c+1}{2}\right) = 0. \end{cases} \tag{3.71}$$

In the following, we shall split the study of this system, as M^3 is given by **Case I** or **Case II** of the classification.

Case I. The system (3.71) is equivalent to the system given by the first three equations of (3.69). Now, M is not minimal if and only if at least one of the components of the mean curvature vector field H does not vanish and, from the first equation of (3.69), it follows that λ^2 must be different from $(c+3)/12$. Thus, again using [10] for the expressions of the curvatures of the three curves, we obtain the first case of the theorem.

Case II. (1) In this case, the second equation of (3.71) is identically satisfied and the other two are equivalent to $c = 5/9$. Thus, from the classification of the integral $\mathcal{C}$-parallel submanifolds, we get the first part of the second case of the theorem.

(2) The second and the third equation of (3.71) are satisfied, in this case, and the first equation is equivalent to

$$3\lambda^4 - 2(c+1)\lambda^2 + \frac{(c+3)^2}{16} = 0.$$

This equation has solutions if and only if

$$c \in \Big(-\infty, \frac{-7 - 8\sqrt{3}}{13}\Big] \cup \Big[\frac{-7 + 8\sqrt{3}}{13}, +\infty\Big),$$

and these solutions are given by

$$\lambda^2 = \frac{4c+4 \pm \sqrt{13c^2+14c-11}}{12}.$$

Since $c > -1/3$ it follows that $c \in [(-7+8\sqrt{3})/13, +\infty)$. Moreover, if $c = 1$, from the above relation, it follows that λ^2 must be equal to 1 or $1/3$, which is a contradiction, and therefore $c \in [(-7+8\sqrt{3})/13, +\infty) \setminus \{1\}$. Further, it is easy to check that $\lambda^2 = (4c+4+\sqrt{13c^2+14c-11})/12 < (c+3)/4$ if and only if $c \in [(-7+8\sqrt{3})/13, 1)$ and $\lambda^2 = (4c+4-\sqrt{13c^2+14c-11})/12 < (c+3)/4$ if and only if $c \in [(-7+8\sqrt{3})/13, +\infty) \setminus \{1\}$. $\square$

3.3.6 Proper-biharmonic submanifolds in the 7-sphere

In this section we shall work with the standard model for simply connected Sasakian space forms $N^7(c)$ with $c+3>0$, which is the unit Euclidean sphere $\mathbb{S}^7$ endowed with its canonical Sasakian structure or with the deformed Sasakian structure introduced by S. Tanno.

In [10] the authors obtained the explicit equation of the 3-dimensional integral $\mathcal{C}$-parallel flat submanifolds in $\mathbb{S}^7(1)$, whilst in [68] we gave the explicit equation of such submanifolds in $\mathbb{S}^7(c)$, $c+3>0$.

Using these results and Theorem 3.53 we easily get the following theorem.

Theorem 3.54 ([67]). *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of $\mathbb{S}^7(c)$, $c = 4/a - 3 > -3$, is proper-biharmonic if and only if either:*

1. *$c > -1/3$ and M^3 is flat, it is locally a product of three curves and its position vector in $\mathbb{C}^4$ is*

$$\begin{aligned} x(u,v,w) = & \frac{\lambda}{\sqrt{\lambda^2+\frac{1}{a}}} \exp\Big(\mathrm{i}\Big(\frac{1}{a\lambda}u\Big)\Big)\mathcal{E}_1 \\ & + \frac{1}{\sqrt{a(\mu-\alpha)(2\mu-\alpha)}} \exp(-\mathrm{i}(\lambda u - (\mu-\alpha)v))\mathcal{E}_2 \\ & + \frac{1}{\sqrt{a\rho_1(\rho_1+\rho_2)}} \exp(-\mathrm{i}(\lambda u + \mu v + \rho_1 w))\mathcal{E}_3 \\ & + \frac{1}{\sqrt{a\rho_2(\rho_1+\rho_2)}} \exp(-\mathrm{i}(\lambda u + \mu v - \rho_2 w))\mathcal{E}_4, \end{aligned}$$

where $\rho_{1,2} = (\sqrt{4\mu(2\mu-\alpha)+\delta^2} \pm \delta)/2$ and $\lambda, \alpha, \mu, \delta$ are real constants given by (3.69) *such that $-1/\sqrt{a} < \lambda < 0$, $0 < \alpha \leq (\lambda^2 - 1/a)/\lambda$, $\alpha \geq \delta \geq 0$, $\alpha > 2\mu$, $\lambda^2 \neq 1/(3a)$ and $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\mathbb{C}^4$ with respect to the usual Hermitian inner product*;

or

(2) M^3 is locally isometric to a product $\gamma \times \bar{M}^2$ between a curve and a $\mathcal{C}$-parallel surface of N, and either

(a) *$c = 5/9$, γ is a helix in $\mathbb{S}^7(5/9)$ with curvatures $\kappa_1 = 1/\sqrt{2}$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $\sqrt{3}/2$, or*

(b) *$c \in [(-7+8\sqrt{3})/13, +\infty)\backslash\{1\}$, γ is a helix in $\mathbb{S}^7(c)$ with curvatures $\kappa_1 = (\lambda^2 - (c+3)/4)/\lambda$ and $\kappa_2 = 1$, and $\bar{M}^2$ is locally isometric to the 2-dimensional Euclidean sphere with radius $2/\sqrt{4\lambda^2 + c + 3}$, where*

$$\lambda < 0 \quad \textit{and} \quad \lambda^2 = \begin{cases} \dfrac{4c + 4 \pm \sqrt{13c^2 + 14c - 11}}{12} & \textit{if} \quad c < 1, \\ \dfrac{4c + 4 - \sqrt{13c^2 + 14c - 11}}{12} & \textit{if} \quad c > 1. \end{cases}$$

Now, applying this theorem in the case of the 7-sphere endowed with its canonical Sasakian structure we get the following Corollary, which also shows that, for $c = 1$, the system (3.69) can be completely solved.

Corollary 3.55 ([67]). *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of $\mathbb{S}^7(1)$ is proper-biharmonic if and only if it is flat, it is locally a product of three curves and its position vector in $\mathbb{C}^4$ is*

$$\begin{aligned} x(u, v, w) = &- \frac{1}{\sqrt{6}} \exp(-\mathrm{i}\sqrt{5}u)\mathcal{E}_1 + \frac{1}{\sqrt{6}} \exp\Big(\mathrm{i}\Big(\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v\Big)\Big)\mathcal{E}_2 \\ &+ \frac{1}{\sqrt{6}} \exp\Big(\mathrm{i}\Big(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_3 \\ &+ \frac{1}{\sqrt{2}} \exp\Big(\mathrm{i}\Big(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_4, \end{aligned}$$

where $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\mathbb{C}^4$ with respect to the usual Hermitian inner product. Moreover, the x_u-curve is a helix with curvatures $\kappa_1 = 4\sqrt{5}/5$ and $\kappa_2 = 1$, the x_v-curve is a helix of order 4 with curvatures $\kappa_1 = \sqrt{29}/\sqrt{10}$, $\kappa_2 = 9\sqrt{2}/\sqrt{145}$ and $\kappa_3 = 2\sqrt{3}/\sqrt{145}$ and the x_w-curve is a helix of order 4 with curvatures $\kappa_1 = \sqrt{5}/\sqrt{2}$, $\kappa_2 = 2\sqrt{3}/\sqrt{10}$ and $\kappa_3 = \sqrt{3}/\sqrt{10}$.

Proof. Since $c = 1$ the system (3.69) becomes

$$\begin{cases} (3\lambda^2 - 1)^2(\lambda^2 - 1) + \lambda^4((\alpha + \mu)^2 + \delta^2) = 0, \\ (\alpha + \mu)(5\lambda^2 + \alpha^2 + \mu^2 - 3) + \mu\delta^2 = 0, \\ \delta(5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 3) = 0, \\ \lambda^2 + \alpha\mu - \mu^2 + 1 = 0 \end{cases} \tag{3.72}$$

with the supplementary conditions

$$-1 < \lambda < 0, \quad 0 < \alpha \leq \frac{\lambda^2 - 1}{\lambda}, \quad \alpha \geq \delta \geq 0, \quad \alpha > 2\mu \quad \text{and} \quad \lambda^2 \neq \frac{1}{3}. \tag{3.73}$$

We note that, since $\alpha > 2\mu$, from the fourth equation of (3.72) it results that $\mu < 0$.

The third equation of system (3.72) suggests that, in order to solve this system, we need to split our study in two cases as δ is equal to 0 or not.

Case 1: $\delta = 0$**.** In this case the third equation holds whatever the values of λ, α and μ are, and so does the condition $\alpha \geq \delta$. We also note that $\alpha \neq -\mu$, since otherwise, from the first equation, it results $\lambda^2 = 1$ or $\lambda^2 = 1/3$, which are both contradictions.

In the following, we shall look for α of the form $\alpha = \omega\mu$, where $\omega \in (-\infty, 0) \setminus \{-1\}$, since $\alpha > 0$, $\mu < 0$ and $\alpha \neq -\mu$. From the second and the fourth equations of the system we have $\lambda^2 = -(\omega^2 + 3\omega - 2)/((\omega-2)(\omega-3))$, $\mu^2 = 8/((\omega-2)(\omega-3))$ and then $\alpha^2 = 8\omega^2/((\omega-2)(\omega-3))$. Replacing in the first equation, after a straightforward computation, it can be written as

$$\frac{8(\omega+1)^3(1-3\omega)}{(\omega-3)^3(\omega-2)} = 0$$

and its solutions are -1 and $1/3$. But $\omega \in (-\infty, 0) \setminus \{-1\}$ and therefore we conclude that there are no solutions of the system that verify all conditions (3.73) when $\delta = 0$.
Case 2: $\delta > 0$**.** In this case the third equation of (3.72) becomes

$$5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 3 = 0.$$

Now, since $\alpha > 0$ and $\mu < 0$, we can take again $\alpha = \omega\mu$, with $\omega \in (-\infty, 0)$, and then, from the last three equations of the system, we easily get $\lambda^2 = -(\omega^2 + 5\omega + 2)/((\omega-1)(\omega-2))$, $\alpha^2 = 8\omega^3/((\omega-1)^2(\omega-2))$, $\mu^2 = 8\omega/((\omega-1)^2(\omega-2))$ and $\delta^2 = 8(\omega+1)^2/(\omega-1)^2$.
Next, from the first equation of (3.72), after a straightforward computation, one obtains

$$\frac{16(\omega+1)^3(\omega+3)}{(\omega-2)(\omega-1)^3} = 0,$$

whose solutions are -3 and -1. If $\omega = -1$ it follows that $\lambda^2 = 1/3$, which is a contradiction, and therefore we obtain that $\omega = -3$. Hence

$$\lambda^2 = \frac{1}{5}, \quad \alpha^2 = \frac{27}{10}, \quad \mu^2 = \frac{3}{10} \quad \text{and} \quad \delta^2 = 2.$$

As $\lambda < 0$, $\alpha > 0$, $\mu < 0$ and $\delta > 0$ it results that $\lambda = -1/\sqrt{5}$, $\alpha = 3\sqrt{3}/\sqrt{10}$, $\mu = -\sqrt{3}/\sqrt{10}$ and $\delta = \sqrt{2}$. It can be easily seen that also the conditions (3.73) are verified by these values, and then, by the meaning of the first statement of Theorem 3.54, we come to the conclusion. □

Remark 3.56 ([67])**.** A proper-biharmonic compact submanifold M of $\mathbb{S}^n$ of constant mean curvature $|H| \in (0,1)$ is of 2-type and mass-symmetric (see [18, 21]). In our case, the Riemannian immersion x can be written as $x = x_1 + x_2$, where

$$\begin{aligned} x_1(u,v,w) =& \frac{1}{\sqrt{2}} \exp\Big(\mathrm{i}\Big(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_4, \\ x_2(u,v,w) =& -\frac{1}{\sqrt{6}}\exp(-\mathrm{i}\sqrt{5}u)\mathcal{E}_1 + \frac{1}{\sqrt{6}}\exp\Big(\mathrm{i}\Big(\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v\Big)\Big)\mathcal{E}_2 \\ &+ \frac{1}{\sqrt{6}}\exp\Big(\mathrm{i}\Big(\frac{1}{\sqrt{5}}u + \frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_3, \end{aligned}$$

and $\Delta x_1 = 3(1-|H|)x_1 = x_1$, $\Delta x_2 = 3(1+|H|)x_2 = 5x_2$, $|H| = 2/3$. Now, Corollary 3.55 could also be proved by using the main result in [9] and [29, Proposition 4.1].

Remark 3.57 ([67]). By a straightforward computation we can deduce that the map x factorizes to a map from the torus $\mathcal{T}^3 = \mathbb{R}^3/\Lambda$ into $\mathbb{R}^8$, where Λ is the lattice generated by the vectors $a_1 = (6\pi/\sqrt{5}, \sqrt{3}\pi/\sqrt{10}, \pi/\sqrt{2})$, $a_2 = (0, -3\sqrt{5}\pi/\sqrt{6}, -\pi/\sqrt{2})$ and $a_3 = (0, 0, -4\pi/\sqrt{2})$, and the quotient map is a Riemannian immersion.

By the meaning of Theorem 3.22 we know that the cylinder over x, given by

$$F(t, u, v, w) = \phi_t(x(u, v, w)),$$

is a proper-biharmonic map into $\mathbb{S}^7(1)$. Moreover, we have the following proposition.

Proposition 3.58 ([67]). *The cylinder over x determines a proper-biharmonic Riemannian embedding from the torus $\mathcal{T}^4 = \mathbb{R}^4/\Lambda$ into $\mathbb{S}^7$, where the lattice Λ is generated by $a_1 = (2\pi/\sqrt{6}, 0, 0, 0)$, $a_2 = (0, 2\pi/\sqrt{6}, 0, 0)$, $a_3 = (0, 0, 2\pi/\sqrt{6}, 0)$ and $a_4 = (0, 0, 0, 2\pi/\sqrt{2})$. The image of this embedding is the standard (extrinsec) product between a Euclidean circle of radius $1/\sqrt{2}$ and three other Euclidean circles, each of radius $1/\sqrt{6}$.*

Proof. As the flow of the characteristic vector field ξ is given by $\phi_t(z) = \exp(-\mathrm{i}t)z$ we get

$$\begin{aligned} F(t,u,v,w) = &-\frac{1}{\sqrt{6}}\exp(-\mathrm{i}(t+\sqrt{5}u))\mathcal{E}_1 + \frac{1}{\sqrt{6}}\exp\Big(\mathrm{i}\Big(-t+\frac{1}{\sqrt{5}}u - \frac{4\sqrt{3}}{\sqrt{10}}v\Big)\Big)\mathcal{E}_2 \\ &+ \frac{1}{\sqrt{6}}\exp\Big(\mathrm{i}\Big(-t+\frac{1}{\sqrt{5}}u+\frac{\sqrt{3}}{\sqrt{10}}v - \frac{3\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_3 \\ &+ \frac{1}{\sqrt{2}}\exp\Big(\mathrm{i}\Big(-t+\frac{1}{\sqrt{5}}u+\frac{\sqrt{3}}{\sqrt{10}}v + \frac{\sqrt{2}}{2}w\Big)\Big)\mathcal{E}_4, \end{aligned}$$

where $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\mathbb{C}^4$ with respect to the usual Hermitian inner product.

Now, we consider the following two orthogonal transformations of $\mathbb{R}^4$:

$$\begin{cases} \frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{10}}u + \frac{\sqrt{3}}{2\sqrt{5}}v + \frac{1}{2}w = t', \\ \frac{2}{\sqrt{5}}u - \frac{\sqrt{6}}{4\sqrt{5}}v - \frac{\sqrt{2}}{4}w = u', \\ \frac{\sqrt{5}}{2\sqrt{2}}v - \frac{\sqrt{3}}{2\sqrt{2}}w = v', \\ \frac{1}{\sqrt{2}}t - \frac{1}{\sqrt{10}}u - \frac{\sqrt{3}}{2\sqrt{5}}v - \frac{1}{2}w = w', \end{cases}$$

and

$$\begin{cases} \frac{\sqrt{2}}{\sqrt{6}}t' + \frac{2}{\sqrt{6}}u' = \widetilde{t}, \\ -\frac{\sqrt{2}}{\sqrt{6}}t' + \frac{1}{\sqrt{6}}u' - \frac{\sqrt{3}}{\sqrt{6}}v' = \widetilde{u}, \\ -\frac{\sqrt{2}}{\sqrt{6}}t' + \frac{1}{\sqrt{6}}u' + \frac{\sqrt{3}}{\sqrt{6}}v' = \widetilde{v}, \\ w' = \widetilde{w}. \end{cases}$$

Then we obtain

$$\begin{aligned}\widetilde{F}(\widetilde{t},\widetilde{u},\widetilde{v},\widetilde{w}) = &-\frac{1}{\sqrt{6}}\exp(-\mathrm{i}(\sqrt{6}\widetilde{t}))\mathcal{E}_1+\frac{1}{\sqrt{6}}\exp(\mathrm{i}(\sqrt{6}\widetilde{u}))\mathcal{E}_2+\frac{1}{\sqrt{6}}\exp(\mathrm{i}(\sqrt{6}\widetilde{v}))\mathcal{E}_3\\&+\frac{1}{\sqrt{2}}\exp(\mathrm{i}(\sqrt{2}\widetilde{w}))\mathcal{E}_4,\end{aligned}$$

which ends the proof. □

Remark 3.59 ([67]). We see that F can be written as $F = F_1+F_2$, where $F_1(t,u,v,w) = \exp(-\mathrm{i}t)x_1$, $F_2(t,u,v,w) = \exp(-\mathrm{i}t)x_2$, and $\Delta F_1 = 2F_1$, $\Delta F_2 = 6F_2$, the mean curvature of y being equal to $1/2$.

Remark 3.60 ([67]). It is known that the parallel flat $(n+1)$-dimensional compact anti-invariant submanifolds in $\mathbb{S}^{2n+1}(1)$ are standard products of circles of radii r_i, $i = 1,\dots,n+1$, where $\sum_{i=1}^{n+1} r_i^2 = 1$ (see [135]). The biharmonicity of such submanifolds was solved in [139].

3.3.7 Proper-biharmonic parallel Lagrangian submanifolds of $\mathbb{C}P^3$

We consider the Hopf fibration $\pi : \mathbb{S}^{2n+1}(1) \to \mathbb{C}P^n(4)$, and $\overline{M}$ a Lagrangian submanifold of $\mathbb{C}P^n$. Then $\widetilde{M} = \pi^{-1}(\overline{M})$ is an $(n+1)$-dimensional anti-invariant submanifold of $\mathbb{S}^{2n+1}$ invariant under the flow-action of the characteristic vector field ξ_0 and, locally, $\widetilde{M}$ is isometric to $\mathbb{S}^1 \times M^n$. The submanifold $\overline{M}$ is a parallel Lagrangian submanifold if and only if M is an integral $\mathcal{C}$-parallel submanifold (see [101]), and it was proved in [66] that a parallel Lagrangian submanifold $\overline{M}$ is biharmonic if and only if M is (-4)-biharmonic.

We recall here that a map $\psi : (M,g) \to (N,h)$ is (-4)-*biharmonic* if it is a critical point of the (-4)-bienergy $E_2(\psi) - 4E(\psi)$, i.e., ψ verifies $\tau_2(\psi) + 4\tau(\psi) = 0$. Also, a real submanifold $\overline{M}$ of $\mathbb{C}P^n$ is called *Lagrangian* if it has dimension n and the complex structure $\overline{J}$ of $\mathbb{C}P^n$ maps the tangent space to $\overline{M}$ onto the normal one.

Thus, in order to determine all proper-biharmonic parallel Lagrangian submanifolds of $\mathbb{C}P^3$, we shall determine the (-4)-biharmonic integral $\mathcal{C}$-parallel submanifolds of $\mathbb{S}^7(1)$.

Just as in the case of Theorem 3.45 we obtain the following theorem.

Theorem 3.61 ([67]). *The integral submanifold* $\mathbf{i} : M^3 \to \mathbb{S}^7(1)$ *is* (-4)-*biharmonic if and only if*

$$\begin{cases}\Delta^{\perp}H + \operatorname{trace} B(\cdot, A_H\cdot) - 7H = 0\\ 4\operatorname{trace} A_{\nabla^{\perp}_{(\cdot)}H}(\cdot) + 3\operatorname{grad}(|H|^2) = 0.\end{cases}$$

Therefore it follows the next proposition.

Proposition 3.62 ([67]). *A non-minimal integral* $\mathcal{C}$-*parallel submanifold* M^3 *of* $\mathbb{S}^7(1)$ *is* (-4)-*biharmonic if and only if*

$$\operatorname{trace} B(\cdot, A_H\cdot) = 6H. \tag{3.74}$$

Now, we can state the theorem.

Theorem 3.63 ([67]). *A 3-dimensional integral $\mathcal{C}$-parallel submanifold M^3 of $\mathbb{S}^7(1)$ is (-4)-biharmonic if and only if either:*

1. *M^3 is flat and it is locally a product of three curves:*
   - *a helix with curvatures $\kappa_1 = (\lambda^2 - 1)/\lambda$ and $\kappa_2 = 1$,*
   - *a helix of order 4 with curvatures $\kappa_1 = \sqrt{\lambda^2 + \alpha^2}$, $\kappa_2 = (\alpha/\kappa_1)\sqrt{\lambda^2 + 1}$ and $\kappa_3 = -(\lambda/\kappa_1)\sqrt{\lambda^2 + 1}$,*
   - *a helix of order 4 with curvatures $\kappa_1 = \sqrt{\lambda^2 + \mu^2 + \delta^2}$, $\kappa_2 = (\delta/\kappa_1)\sqrt{\lambda^2 + \mu^2 + 1}$ and $\kappa_3 = (\kappa_2/\delta)\sqrt{\lambda^2 + \mu^2}$, if $\delta \neq 0$, or a circle with curvature $\kappa_1 = \sqrt{\lambda^2 + \mu^2}$, if $\delta = 0$,*

   where $\lambda, \alpha, \mu, \delta$ are constants given by

$$\begin{cases} (3\lambda^2 - 1)(3\lambda^4 - 8\lambda^2 + 1) + \lambda^4((\alpha + \mu)^2 + \delta^2) = 0, \\ (\alpha + \mu)(5\lambda^2 + \alpha^2 + \mu^2 - 7) + \mu\delta^2 = 0, \\ \delta(5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 7) = 0, \\ 1 + \lambda^2 + \alpha\mu - \mu^2 = 0 \end{cases} \tag{3.75}$$

   such that $-1 < \lambda < 0$, $0 < \alpha \leq (\lambda^2 - 1)/\lambda$, $\alpha \geq \delta \geq 0$, $\alpha > 2\mu$ and $\lambda^2 \neq 1/3$;

or

(2) M^3 is locally isometric to a product $\gamma \times \bar{M}^2$ between a helix with curvatures $\kappa_1 = (\sqrt{13} - 1)/\sqrt{12 - 3\sqrt{13}}$ and $\kappa_2 = 1$, and a $\mathcal{C}$-parallel surface of $\mathbb{S}^7(1)$ which is locally isometric to the 2-dimensional Euclidean sphere with radius $\sqrt{3/(7 - \sqrt{13})}$.

Proof. It is easy to see that the equation (3.74) is equivalent to the system

$$\begin{cases} \Big(\sum_{i=1}^{3} \lambda_i\Big)\Big(\sum_{i=1}^{3} \lambda_i^2 - 6\Big) + (\alpha + \mu)(\alpha\lambda_2 + \mu\lambda_3) + (\beta + \delta)(\beta\lambda_2 + \delta\lambda_3) = 0, \\ \Big(\sum_{i=1}^{3} \lambda_i\Big)(\alpha\lambda_2 + \mu\lambda_3) + (\alpha + \mu)(2\lambda_2^2 + \alpha^2 + 3\beta^2 + \mu^2 + \beta\delta - 6) + \mu(\beta + \delta)^2 = 0, \\ \Big(\sum_{i=1}^{3} \lambda_i\Big)(\beta\lambda_2 + \delta\lambda_3) + \beta(\alpha + \mu)^2 + (\beta + \delta)(2\lambda_3^2 + \delta^2 + 3\mu^2 + \beta^2 + \alpha\mu - 6) = 0. \end{cases} \tag{3.76}$$

In the same way as for the study of biharmonicity, we shall split the study of this system, as M^3 is given by **Case I** or **Case II** of the classification.

Case I. The system (3.76) is equivalent to the system given by the first three equations of (3.75) and, just like in the proof of Theorem 3.53, we conclude the result.

Case II. (1) It is easy to verify that this case cannot occur in this setting.

(2) The second and the third equation of system (3.76) are satisfied and the first equation is equivalent to $3\lambda^4 - 8\lambda^2 + 1 = 0$, whose solutions are $\lambda^2 = (4 \pm \sqrt{13})/3$. Since $\lambda^2 < 1$ it follows that $\lambda^2 = (4 - \sqrt{13})/3$ and this, together with the classification of the integral $\mathcal{C}$-submanifolds, leads to the conclusion. □

Using the explicit equation of the 3-dimensional integral $\mathcal{C}$-parallel flat submanifolds in $\mathbb{S}^7(1)$ (see [10]), we obtain the following corollary.

Corollary 3.64 ([67]). *Any 3-dimensional flat (-4)-biharmonic integral $\mathcal{C}$-parallel submanifold M^3 of $\mathbb{S}^7(1)$ is given locally by*

$$\begin{aligned} x(u,v,w) =& \frac{\lambda}{\sqrt{\lambda^2+1}} \exp\Big(\mathrm{i}\Big(\frac{1}{\lambda}u\Big)\Big)\mathcal{E}_1 + \frac{1}{\sqrt{(\mu-\alpha)(2\mu-\alpha)}} \exp(-\mathrm{i}(\lambda u - (\mu-\alpha)v))\mathcal{E}_2 \\ &+ \frac{1}{\sqrt{\rho_1(\rho_1+\rho_2)}} \exp(-\mathrm{i}(\lambda u + \mu v + \rho_1 w))\mathcal{E}_3 \\ &+ \frac{1}{\sqrt{\rho_2(\rho_1+\rho_2)}} \exp(-\mathrm{i}(\lambda u + \mu v - \rho_2 w))\mathcal{E}_4, \end{aligned}$$

where $\rho_{1,2} = (\sqrt{4\mu(2\mu-\alpha)+\delta^2} \pm \delta)/2$, $-1 < \lambda < 0$, $0 < \alpha \le (\lambda^2-1)/\lambda$, $\alpha \ge \delta \ge 0$, $\alpha > 2\mu$, $\lambda^2 \ne 1/3$, the tuple $(\lambda, \alpha, \mu, \delta)$ being one of the following

$$\Bigg(-\sqrt{\frac{4-\sqrt{13}}{3}}, \sqrt{\frac{7-\sqrt{13}}{6}}, -\sqrt{\frac{7-\sqrt{13}}{6}}, 0\Bigg),$$

$$\Bigg(-\sqrt{\frac{1}{5+2\sqrt{3}}}, \sqrt{\frac{45+21\sqrt{3}}{13}}, -\sqrt{\frac{6}{21+11\sqrt{3}}}, 0\Bigg),$$

or

$$\Bigg(-\sqrt{\frac{1}{6+\sqrt{13}}}, \sqrt{\frac{523+139\sqrt{13}}{138}}, -\sqrt{\frac{79-17\sqrt{13}}{138}}, \sqrt{\frac{14+2\sqrt{13}}{3}}\Bigg),$$

and $\{\mathcal{E}_i\}_{i=1}^4$ is an orthonormal basis of $\mathbb{C}^4$ with respect to the usual Hermitian inner product.

Proof. In order to solve the system (3.75), we first note that, since $\alpha > 2\mu$, from the fourth equation it results $\mu < 0$.

The third equation suggests that we need to split our study in two cases as δ is equal to 0 or not.

Case 1: $\delta = 0$. In this case the third equation holds whatever the values of λ, α and μ are, and so does the condition $\alpha \ge \delta$.

If $\alpha = -\mu$ we easily obtain that the solution of the system is

$$\lambda = -\sqrt{\frac{4-\sqrt{13}}{3}}, \quad \alpha = \sqrt{\frac{7-\sqrt{13}}{6}}, \quad \mu = -\sqrt{\frac{7-\sqrt{13}}{6}}.$$

In the following, we shall look for α of the form $\alpha = \omega\mu$, where $\omega \in (-\infty, 0) \setminus \{-1\}$, since $\alpha > 0$ and $\mu < 0$. From the second and the fourth equations of the system we have $\lambda^2 = -(\omega^2 + 7\omega - 6)/((\omega-2)(\omega-3))$, $\mu^2 = 12/((\omega-2)(\omega-3))$ and then $\alpha^2 = 12\omega^2/((\omega-2)(\omega-3))$. Replacing in the first equation, after a straightforward computation, it can be written as

$$3\omega^6 + 16\omega^5 - 58\omega^4 - 140\omega^3 + 531\omega^2 - 444\omega + 108 = 0,$$

which is equivalent to

$$(\omega - 2)^2(3\omega^4 + 28\omega^3 + 42\omega^2 - 84\omega + 27) = 0,$$

whose solutions are 2, $-3 \pm 2\sqrt{3}$ and $(-5 \pm 2\sqrt{13})/3$. From these solutions the only one to verify the supplementary conditions is $\omega = -3 - 2\sqrt{3}$, for which we have

$$\lambda = -\sqrt{\frac{1}{5 + 2\sqrt{3}}}, \quad \alpha = \sqrt{\frac{45 + 21\sqrt{3}}{13}}, \quad \mu = -\sqrt{\frac{6}{21 + 11\sqrt{3}}}.$$

Case 2: $\delta > 0$. In this case the third equation of (3.75) becomes

$$5\lambda^2 + \delta^2 + 3\mu^2 + \alpha\mu - 7 = 0.$$

Now, again taking $\alpha = \omega\mu$, this time with $\omega \in (-\infty, 0)$, from the last three equations of the system, we easily get

$$\lambda^2 = -\frac{\omega^2 + 9\omega + 2}{(\omega - 1)(\omega - 2)}, \quad \alpha^2 = \frac{12\omega^3}{(\omega - 1)^2(\omega - 2)},$$

$$\mu^2 = \frac{12\omega}{(\omega - 1)^2(\omega - 2)}, \quad \delta^2 = \frac{12(\omega + 1)^2}{(\omega - 1)^2}.$$

Replacing in the first equation of the system we obtain the solutions $-2 \pm \sqrt{3}$ and $-4 \pm \sqrt{13}$, from which only $\omega = -4 - \sqrt{13}$ verifies the supplementary conditions. Therefore, we obtain

$$\lambda = -\sqrt{\frac{1}{6 + \sqrt{13}}}, \quad \alpha = \sqrt{\frac{523 + 139\sqrt{13}}{138}}, \quad \mu = -\sqrt{\frac{79 - 17\sqrt{13}}{138}}, \quad \delta = \sqrt{\frac{14 + 2\sqrt{13}}{3}},$$

and we are done. □

Remark 3.65 ([67])**.** By a straightforward computations we can check that the images of the cylinders over the above x are, respectively: the standard (extrinsic) product of a circle of radius $\sqrt{(5 - \sqrt{13})/12}$ and three circles, each of radius $\sqrt{(7 + \sqrt{13})/36}$; the standard product of two circles each of radius $\sqrt{(3 + \sqrt{3})/12}$ and two circles each of radius $\sqrt{(3 - \sqrt{3})/12}$; the standard product of a circle of radius $\sqrt{(5 + \sqrt{13})/12}$ and three circles each of radius $\sqrt{(7 - \sqrt{13})/36}$.

Further developments

In [72], the authors studied proper-biharmonic submanifolds in $\mathbb{S}^n \times \mathbb{R}$. First, they gave a Simons type formula for submanifolds with parallel mean curvature vector field (PMC) and then, the authors obtained a gap theorem for the mean curvature of certain complete PMC proper-biharmonic submanifolds. Moreover, the complete determination of all PMC proper-biharmonic surfaces in $\mathbb{S}^n \times \mathbb{R}$ was obtained. A first research direction would be the continuation of the study of proper-biharmonic submanifolds in $\mathbb{S}^n \times \mathbb{R}$. In particular, it would be interesting to determine all proper-biharmonic surfaces in $\mathbb{S}^2 \times \mathbb{R}$ without any additional hypothesis.

Recently, the bi-conservative submanifolds have been introduced in [31]. By definition, such a submanifold has free divergence bi-tensor field, i.e. $\operatorname{div} S_2 = 0$, and it represents a generalization of the H-hypersurfaces in Euclidean spaces $\mathbb{R}^n$. These hypersurfaces were introduced by T. Hasanis and T. Vlachos in [75]. One can prove that a submanifold is bi-conservative if and only if the tangent part of the bitension field vanishes. We intend to continue the study of the bi-conservative submanifolds. First, we shall study the bi-conservative surfaces in 4-dimensional Euclidean space $\mathbb{R}^4$, and then we shall study the bi-conservative submanifolds in real space forms, especially in $\mathbb{S}^n$.

Recently, H. Urakawa has studied for the first time the biharmonic maps with values in a compact Lie group endowed with a bi-invariant metric ([129]). By using the Maurer-Cartan form, the author gave the characterization for the biharmonic maps defined on an open domain in $\mathbb{R}^2$, endowed with a conformal metric to the usual one, and with values in a compact Lie group endowed with a bi-invariant metric. Another result that he obtained is the explicit determination of biharmonic maps from the real line into the group $SU(2)$. A research direction will be the study of biharmonic maps (with additional properties) in Lie groups.

The study of the biharmonicity of vector fields, thought of as maps from the base manifold (M, g) to its tangent bundle (TM, G), where (M, g) is a compact Lie group with a bi-invariant metric and G is the corresponding Sasaki metric, was recently initiated in [93]. Another research direction will be the study of the biharmonicity of vector fields when the domain manifold is a (non-compact) Lie group.

Although the theory of biharmonic maps and submanifolds is the main topic of investigation, we shall further extend the search for, and the analysis of other similar problems, also discussing their applications in other theories. There are various fourth-order elliptic equations similar to the biharmonic one, some of them derived from Geometry, but also as mathematical models in Mechanics or Physics. An example derived from Geometry is the equation of motion for the constrained Lagrangian associated to a non-holonomic Lagrangian of second order. This equation is known in literature as the Heisenberg spinning particle equation. An example of a fourth-order elliptic equation that appears in Sciences is the steady state of a curvature-drive flow model for two dimensional biphasic biological systems, such as the immunological synapse. A research direction will be the study the biharmonic and other similar equations from the analytical point of view, by investigating some properties of their solutions. We expect to obtain existence results, conditions for the periodicity of the solutions, and stability properties.

Some of the results concerning the theory of biharmonic maps and submanifolds could be the main topic of a course for PhD students and, mixed with elements from the theory of harmonic maps, of a graduate course. Considering that this theory is mainly based on classical, very beautiful results of Differential Geometry, we expect that such a course will be very appealing and useful for young researchers. The very proof of this claim is the activity of our younger collaborators: Adina Balmuş and Dorel Fetcu.

We should mention that well known mathematicians like Paul Baird, Bang-Yen Chen, Eric Loubeau, Stefano Montaldo, Ye-Lin Ou, Harold Rosenberg, Hajime Urakawa are interested in this kind of problems and some of their PhD. students are preparing (or have already defended) their theses on biharmonicity.

Another way to attract young researchers to this field of Riemannian Geometry and Geometric Analysis, is to include them in national and international research grants. The author of this thesis has a good experience getting such grants, as he was awarded 6 national competitive grants, as well as more than 10 international fellowships.

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