Abstract

In this thesis we present our recent works on analysis of gamma function and related functions. The results obtained (collected from 60 ISI papers and 80 papers published in journals indexed by ZMath or MR) are in the theory of asymptotic analysis, approximation of gamma and polygamma functions, or the theory of completely monotonic functions.

In asymptotic analysis, deriving an asymptotic expansion is considered to be technically difficult. Classical construction of asymptotic series involving gamma, polygamma and related functions makes appeal to Bernoulli numbers and other non elementary notions.

The initial motivation of this group of research papers put toghether in this thesis is the work C. Mortici [Product Approximations via Asymptotic Integration Amer. Math. Monthly 117 (2010) 434-441] where a simple strategy for constructing asymptotic series is introduced.

This method is based on the natural idea that when the above series is truncated at the mth term, the approximation obtained should be the most precise possible. The method presented here is quite elementary, consisting only in developing some functions in power series, and then solving a linear, triangular system.

There are many situations of practical problems from pure mathematics, or other branches of science when we are forced to deal with large factorials. In consequence, the factorial function defined for non-negative integers, and its extension - gamma function - to the real and complex values, except the negative integers, have caught the interest of many authors.

As a direct computation cannot be made even by the computer programs, approximation formulas were constructed, one of the most known and most used being the Stirling's formula. If in probability theory, or statistics, approximations like Stirling are satisfactory, in pure mathematics, more precise estimates are required. In this sense, we present a group of original results which extend other recent results regarding famous formulas due to Stirling, Burnside, Gosper, or introduce new class of estimates.

The research continues to the problem of constructing asymptotic series corresponding to some approximation formulas. Having in mind an excellent result of H. Alzer in [On some inequalities for the gamma and psi functions, Math. Comp., 66 (1997), no. 217, 373-389], about positioning of an expression involving gamma function between consecutive truncations of the associated asymptotic series, we obtain new inequalities and estimates for gamma and polygamma functions. First note that we introduced and proved a generalization of Alzer inequality for a family of approximations containing Stirling and Burnside formulas.

These inequalities and estimates are established and proved using the theory of completely monotonic functions. Completely monotonic functions involving the logarithm of gamma function are important because they produce sharp

bounds for the function itself and for their derivatives. We exploit the advantage of integral representations of polygamma functions, while the key proof in obtaining such results is the famous Hausdorff-Bernstein-Widder theorem which states that a function is completely monotonic if and only if it is a Laplace transform.

We dedicate a section of this thesis to the Ramanujan formula for gamma function, where important results were published in highly rated journals, such as Ramanujan Journal, or Applied Mathematics and Computation. Monotonicity results, or new more accurate approximations are stated, including formulas involving rooths of order eight, ten and twelve. Moreover, a general method for establishing such formulas of every even order is described.

The next section is dedicated to a class of results about the quotient of gamma functions. The problem of estimating the ratio of gamma functions in terms of digamma function is important since it is related to inequalities of Kershaw-Gautschi type. Also in this problem we obtained estimates of ratio of gamma functions using asymptotic analysis theory and the theory of completely monotonic functions. Numerical computations were made using computer softwares which were of great help in extending or establishing new results.

Finally we show how using similar methods, results in computational number theory can be refined or other new formulas can be stated. New classes of sequences convergent to Euler's constant are investigated. Special choices of parameters show that the new families defined include the original sequence defined by Euler, as well as more recently defined. It is shown how the rate of convergence of the sequences can be improved by computing optimal values of the parameters. Having in mind that an important concern in the theory of mathematical constants is the definition of new sequences which converge with increasingly higher speed, we concentrate to obtain new accurate estimates for Somos' constants, sharp bounds for the Apery's constant, Catalan constant and of course Euler-Mascheroni constant.

In conclusion, starting from a method of construction of asymptotic series, we have gradually obtained new results in asymptotic theory, computational number theory, approximation theory, inequalities, completely monotonic functions theory, with further possible applications in probabilities, statistics, statistical physics and other branches of science.